An Introduction to Portfolio Theory

John Norstad

j-norstad@northwestern.edu
http://www.norstad.org

April 10, 1999
Updated: November 3, 2011

Abstract

We introduce the basic concepts of portfolio theory, including the notions of efficiency, risk/return graphs, the efficient frontier, iso-utility (indifference) curves, and asset allocation optimization problems. We develop the theory in both a simplified setting where we assume that returns are normally distributed and in the more palatable random walk model where returns are lognormally distributed.

We assume that the reader is familiar with the material presented in references [4, 5, 6].

Contents

1 Introduction **2**

2 Efficiency **3**

3 Normal Returns **4**
- 3.1 Theorems and Proofs . . . 4
- 3.2 Risk/Return Graphs . . . 8
- 3.3 Examples . . . 9
  - 3.3.1 Example 1 – Bonds and Stocks . . . 9
  - 3.3.2 Example 2 – Cash, Bonds and Stocks . . . 14

4 Lognormal Returns **16**
- 4.1 Theorems and Proofs . . . 16
- 4.2 Risk/Return Graphs . . . 22
- 4.3 Examples . . . 23
  - 4.3.1 Example 1 – Bonds and Stocks . . . 23
  - 4.3.2 Example 2 – Cash, Bonds and Stocks . . . 23

List of Figures

1 Risk/Return Graphs . . . 8
2 Example 1 – Bonds and Stocks . . . 9
3 $A = 4$ Iso-Utility Curves . . . 11
4 $A = 2$ Iso-Utility Curves . . . 13
5 Example 2 – Cash, Bonds and Stocks . . . 14
6 Risk/Return Graphs . . . 22
7 Example 1 – Bonds and Stocks . . . 24
8 $A = 4$ Iso-Utility Curves . . . 25
9 $A = 2$ Iso-Utility Curves . . . 26
10 Example 2 – Cash, Bonds and Stocks . . . 27

1 Introduction

Portfolio theory was first discovered and developed by Harry Markowitz in the 1950's.[1] His work forms the foundation of modern Finance. The resulting theory as modified and extended by many researchers is often called "modern portfolio theory."

In portfolio theory it is often assumed for the sake of simplicity that returns are normally distributed over the time period under analysis.[2] With this assumption, portfolio efficiency is determined by simply compounded expected returns and the standard deviations of the simply compounded returns. The additional assumption of negative exponential utility leads to portfolio optimization problems that are linear in return and variance. In section 3 we present a rigorous review of this theory.

The assumption of normally distributed returns leads to problems when trying to extend the analysis to longer time periods or to multiple time periods, since long-term returns are far from normally distributed. Indeed, even over a single year, the lognormal distribution implied by the random walk model, while still not perfect, is a much better approximation to the distribution of observed historical returns for common financial assets like stocks and bonds. Lognormal returns are also consistent with the Central Limit Theorem and with limited liability, two theoretical issues which also cause problems if we assume normally distributed returns.

In the random walk model, portfolio efficiency is determined by instantaneous expected returns and the standard deviations of these returns. The additional assumption of iso-elastic utility leads to portfolio optimization problems that are linear in return and variance. In section 4 we develop the theory for the lognormal returns implied by the random walk model.

[1] See reference [2].
[2] See for example Sharpe [7].

2 Efficiency

The notion of "efficiency" is central to portfolio theory and is the main focus of this paper. The formal definition of this notion uses utility theory and assumes that all investors are risk-averse.

Suppose I_1 and I_2 are two investments whose ending values at time t are given by the random variables $w_1(t)$ and $w_2(t)$. We say that I_1 is *more efficient than* I_2 for time horizon t if the expected value of the utility of $w_1(t)$ is greater than the expected value of the utility of $w_2(t)$ for all utility functions:

$$E(U(w_1(t))) > E(U(w_2(t))) \quad \text{for all } U$$

The relation "more efficient than or equal to" defines a partial ordering on any set of feasible investments (over a given time horizon).[3] If an investment I_1 is more efficient than some other investment I_2 then all risk-averse investors prefer I_1 to I_2 (for that time horizon).

Given a set of feasible investment alternatives, a member I of the set is said to be *efficient* for the feasible set or "with respect to" the feasible set if it is maximal with respect to the "more efficient than or equal to" partial ordering on the set. That is, if no other member of the set is more efficient than I.

When selecting among a collection of competing investment alternatives for a given time horizon, it suffices to consider only the efficient investments in the set. Different investors acting under the principle of expected utility maximization with different utility functions may select different efficient investments, but none of them will ever select an inefficient one.

[3]The proof that this relation is reflexive, antisymmetric, and transitive (and is hence a partial ordering) is left to the reader. A better formulation would be to first take the equivalence relation defined by the expected utility being equal for all utility functions, then define a partial ordering on the resulting factor set. Our version is adequate for our needs, however.

3 Normal Returns

3.1 Theorems and Proofs

Lemma 3.1 *For a given time horizon t suppose the return on an investment I is normally distributed with mean r and standard deviation s. Let U be the negative exponential utility function with coefficient of risk aversion $A > 0$: $U(w) = -e^{-Aw}$. Let w_0 be the investor's wealth at the beginning of the time period. Then the expected utility of wealth $w(t)$ at the end of the time period is:*

$$E(U(w(t))) = -e^{-Aw_0(1+r-\frac{1}{2}Aw_0s^2)}$$

Proof:

For the investment I with normally distributed return with mean r and standard deviation s over time horizon t and initial wealth w_0, the ending value $w(t)$ is the normally distributed random variable:

$$w(t) = w_0(sX + 1 + r) \quad \text{where } X \text{ is } N[0,1]$$

$$\begin{aligned}
E(U(w(t))) &= \int_{-\infty}^{\infty} U(w_0(sx+1+r))\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \int_{-\infty}^{\infty} -e^{-Aw_0(sx+1+r)}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= -e^{-Aw_0(1+r)}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2-Aw_0sx}dx \\
&= \text{(complete the square)} \\
&\quad -e^{-Aw_0(1+r)}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(x+Aw_0s)^2/2+\frac{1}{2}A^2w_0^2s^2}dx \\
&= -e^{-Aw_0(1+r)+\frac{1}{2}A^2w_0^2s^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(x+Aw_0s)^2/2}dx \\
&= \text{(substitute } y = x + Aw_0s\text{)} \\
&\quad -e^{-Aw_0(1+r)+\frac{1}{2}A^2w_0^2s^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\
&= -e^{-Aw_0(1+r-\frac{1}{2}Aw_0s^2)}
\end{aligned}$$

Theorem 3.1 *A utility-maximizing investor with initial wealth w_0 and a negative exponential utility function with coefficient of risk aversion A, when faced with a decision among a set of competing feasible investment alternatives F all of whose elements have normally distributed returns, acts to select an investment $I \in F$ which maximizes*

$$r_I - \frac{1}{2}Aw_0 s_I^2$$

where r_I and s_I are the expected return and standard deviation of return for investment I respectively.

Proof:

The investor acts to maximize the expected utility of end-of-period wealth. By Lemma 3.1, this is:

$$-e^{-Aw_0(1+r_I-\frac{1}{2}Aw_0 s_I^2)}$$

This is clearly equivalent to maximizing:

$$r_I - \frac{1}{2}Aw_0 s_I^2$$

Note that for very small values of the coefficient of risk aversion A (near zero), the investor is primarily concerned with maximizing expected return, and has little concern for risk. Conversely, for very large values of A, the investor is primarily concerned with minimizing risk (variance = standard deviation squared).

The multiplier $\frac{1}{2}Aw_0$ measures the rate at which the investor is willing to trade return for risk. For example, if this multiplier is 1, the investor requires an equal increase in expected return for every corresponding increase in variance of equal magnitude.

Note the role played by initial wealth w_0. As the investor's wealth increases, he becomes rapidly more risk-averse, requiring ever greater increased return per unit of increased risk. This increasing relative risk aversion as wealth increases is a characteristic of the negative exponential utility function.

Some theorists using this model of normally distributed returns prefer to use the negative exponential utility function to measure utility as a function of return (relative change in wealth over the period) rather than end-of-period wealth.[4] In this case the initial wealth variable w_0 disappears from the equations. The expected utility becomes $-e^{-A(r-\frac{1}{2}As^2)}$ and utility-maximizing investors act to maximize:

$$r_I - \frac{1}{2}As_I^2$$

In this modification of the theory, the optimal investment is independent of the investor's current wealth. This is similar to, but not identical to, the constant relative risk aversion of the iso-elastic utility of wealth functions.

[4] See Sharpe [7].

Theorem 3.2 *Suppose I_1 and I_2 are two investments with normally distributed returns over time horizon t, with mean returns r_1 and r_2 and standard deviations of returns s_1 and s_2 respectively over time horizon t, and with initial wealth w_0 for both investments. Then I_1 is more efficient than I_2 if and only if:*

$$r_1 \geq r_2 \text{ and } s_1 \leq s_2$$

with strict inequality holding in at least one of the inequalities.

Proof:

For an investment I with normally distributed return with mean r and standard deviation s over time horizon t and initial wealth w_0, the ending value $w(t)$ is the normally distributed random variable:

$$w(t) = w_0(sX + 1 + r) \quad \text{where } X \text{ is } N[0,1]$$

For a given utility function U define:

$$u(r,s) = E(U(w(t))) = \int_{-\infty}^{\infty} U(w_0(sx+1+r)) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

We prove the "if" direction by showing that $\frac{\partial u}{\partial r} > 0$ and $\frac{\partial u}{\partial s} < 0$ for $s > 0$.

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \int_{-\infty}^{\infty} U'(w_0(sx+1+r)) w_0 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx > 0 \quad \text{because } U' > 0 \\
\frac{\partial u}{\partial s} &= \int_{-\infty}^{\infty} U'(w_0(sx+1+r)) w_0 x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_{-\infty}^{0} U'(w_0(sx+1+r)) w_0 x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \\
&\quad \int_{0}^{\infty} U'(w_0(sx+1+r)) w_0 x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_{0}^{\infty} U'(w_0(s(-x)+1+r)) w_0(-x) \frac{1}{\sqrt{2\pi}} e^{-(-x)^2/2} dx + \\
&\quad \int_{0}^{\infty} U'(w_0(sx+1+r)) w_0 x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_{0}^{\infty} [U'(w_0(sx+1+r)) - U'(w_0(-sx+1+r))] w_0 x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
\end{aligned}$$

$U'' < 0$, so U' is a decreasing function, so whenever $a > b$ we must have $U'(a) < U'(b)$ and hence $U'(a) - U'(b) < 0$. Thus, for $x > 0$ and $s > 0$, the expression in square brackets above is negative, and hence the entire integral is negative and we have our result.

Note the intuition behind this first half of our proof. The non-satiation property ($U' > 0$) implies that given a fixed risk (volatility), investors prefer higher returns to lower returns (greater expected end-of-period wealth). The risk aversion property ($U'' < 0$) implies that given a fixed return, investors prefer lower volatility to higher volatility (less risk). This is true for all risk-averse investors, regardless of their particular utility functions.

To prove the "only if" direction it suffices to show that if $r_1 < r_2$ and $s_1 < s_2$ then there is at least one utility function U with $E(U(w_1(t))) < E(U(w_2(t)))$ and at least one other utility function V with $E(V(w_1(t))) > E(V(w_2(t)))$.

We use the negative exponential utility function and Theorem 3.1 in this half of the proof. The basic idea is to find a small enough coefficient of risk aversion A so that the investor will prefer the second investment with the higher return (perhaps only a slightly higher return), even though the second investment also has higher risk (perhaps much higher risk).

By Theorem 3.1, our investor acts to maximize $r - \frac{1}{2}Aw_0s^2$. In order for the second investment to have greater expected utility than the first investment we must have:

$$r_2 - \frac{1}{2}Aw_0s_2^2 > r_1 - \frac{1}{2}Aw_0s_1^2$$

Solve this inequality for A:

$$A < \frac{2}{w_0}\frac{r_2 - r_1}{s_2^2 - s_1^2} > 0 \qquad \text{(because } r_2 > r_1 \text{ and } s_2 > s_1\text{)}$$

Thus, for sufficiently small values of A (which might be very close to zero in extreme cases), the expected utility of the second investment is greater than the expected utility of the first investment.

For large values of A the situation is reversed, and the investor prefers the first investment to the second investment, because the increased return does not adequately compensate for the increased risk.

Thus neither investment is more efficient than the other one. More risk-averse investors prefer the first one because it has lower risk. Less risk-averse investors prefer the second one because it has a higher expected return.

3.2 Risk/Return Graphs

We can visualize Theorem 3.2 by graphing investments in the feasible set with standard deviation s on the x axis and expected return r on the y axis.

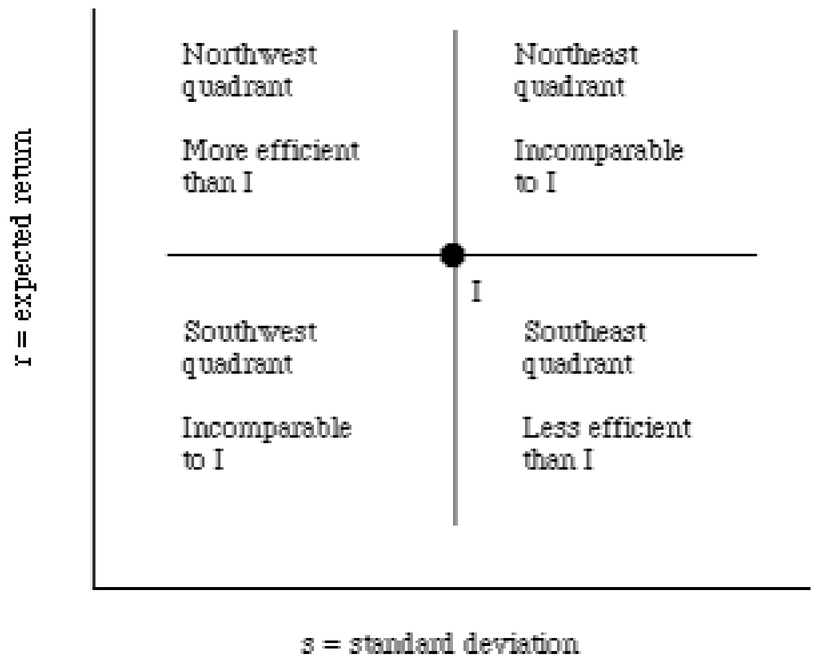

Figure 1: Risk/Return Graphs

For *any* investment J in the northwest quadrant, *all* risk-averse investors prefer J to I.

For *any* investment J in the southeast quadrant, *all* risk-averse investors prefer I to J.

For *any* investment J in the southwest or northeast quadrant, *some* risk-averse investor prefers J to I, and *some other* risk-averse investor prefers I to J.

Note that the horizontal and vertical lines which intersect at investment I belong to the northwest and southeast quadrants. For example, if J lies exactly on the vertical line above I, all risk-averse investors prefer J to I.

3.3 Examples

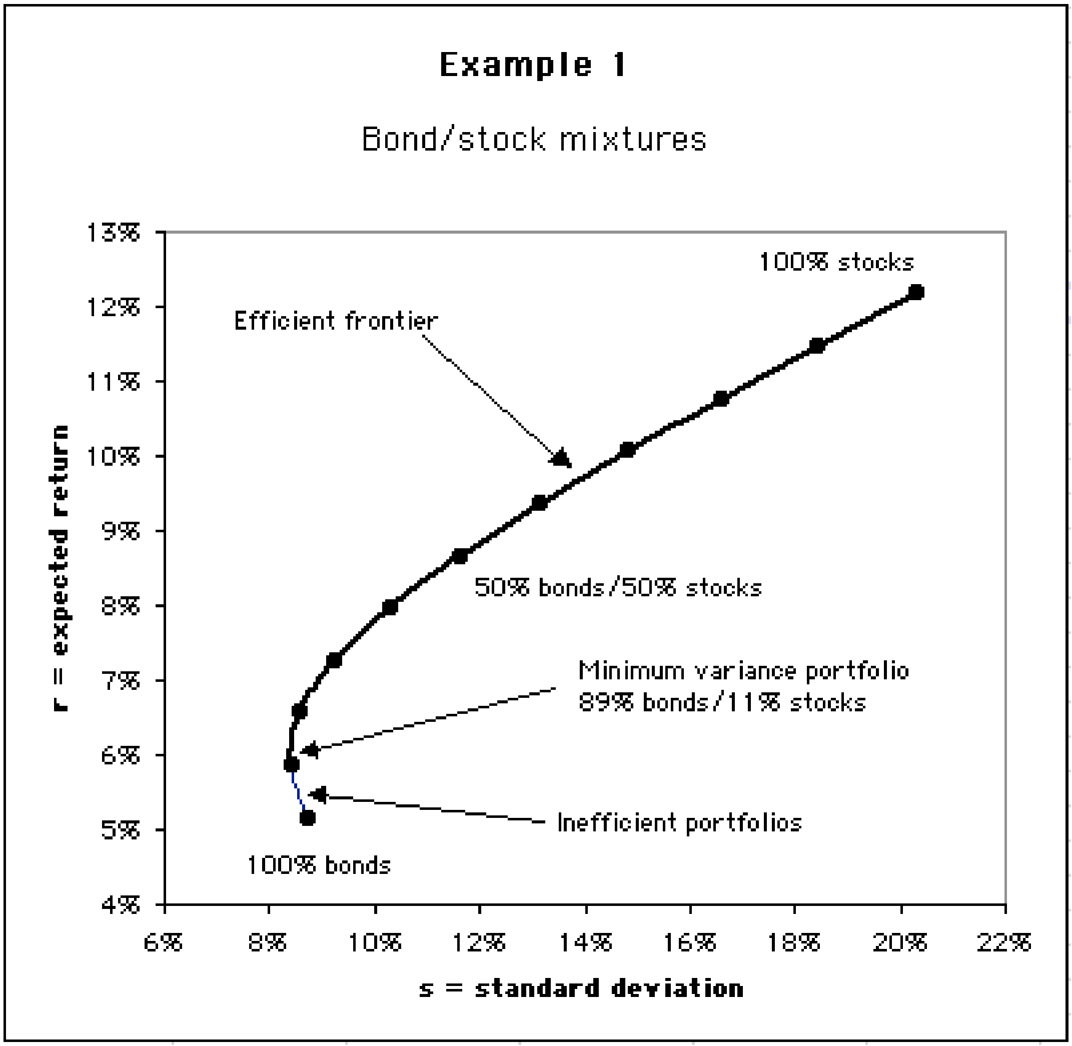

Figure 2: Example 1 – Bonds and Stocks

3.3.1 Example 1 – Bonds and Stocks

As a first example we consider the feasible set of investment alternatives consisting of all portfolios mixing 20 year U.S. Treasury bonds and S&P 500 large U.S. stocks. We assume that leverage and short sales are not allowed, so each portfolio consists of some percentage x of bonds and $100 - x$ of stocks where

$0 \leq x \leq 100$. Our time horizon is one year. We estimate expected returns and standard deviations for these portfolios using historical market return data from 1926 through 1994.[5]

Using historical market return data to estimate future expected returns and standard deviations is a common practice. This is not required by the theory, however. Analysts can and do estimate these parameters using a variety of techniques and forecasting models. Our use of historical data is only an example of one way in which these forecasts might be made.

Figure 2 shows the risk/return graph.

In this example the feasible set plots as a curve. The most conservative portfolio, 100% bonds, has both lower expected return and lower risk than does the most aggressive portfolio, 100% stocks.

The portfolio with the smallest standard deviation (equivalently, the smallest variance), is 89% bonds and 11% stocks. The portfolios which are more conservative than this minimum variance portfolio are inefficient, since the minimum variance portfolio has both a greater expected return and a smaller standard deviation. This phenomenon and the general shape of the curve are typical when a more conservative asset is mixed with a more risky asset.

All of the portfolios which are more aggressive than the minimum variance portfolio are efficient.

The set of all efficient portfolios is called the *efficient frontier*. In general, it consists of the portfolios which lie on the northwest boundary of the feasible set. All risk-averse investors who act to maximize expected utility have an optimal portfolio on this frontier.

Consider an individual investor faced with the problem of selecting a bond/stock asset allocation in this feasible set of investment alternatives. It suffices to consider only the efficient portfolios, so the optimal choice will have at least 11% stocks, even if our investor is very conservative.

Suppose for the sake of example that our investor has a negative exponential utility function measured as a function of return rather than end-of-period wealth, with coefficient of risk aversion $A = 4$. This investor wants to maximize the following function over the feasible set (or, equivalently, over the efficient frontier):

$$r - \frac{1}{2}As^2 = r - 2s^2$$

[5]All of the historical data used in this paper is from Table 2-4 in [1].

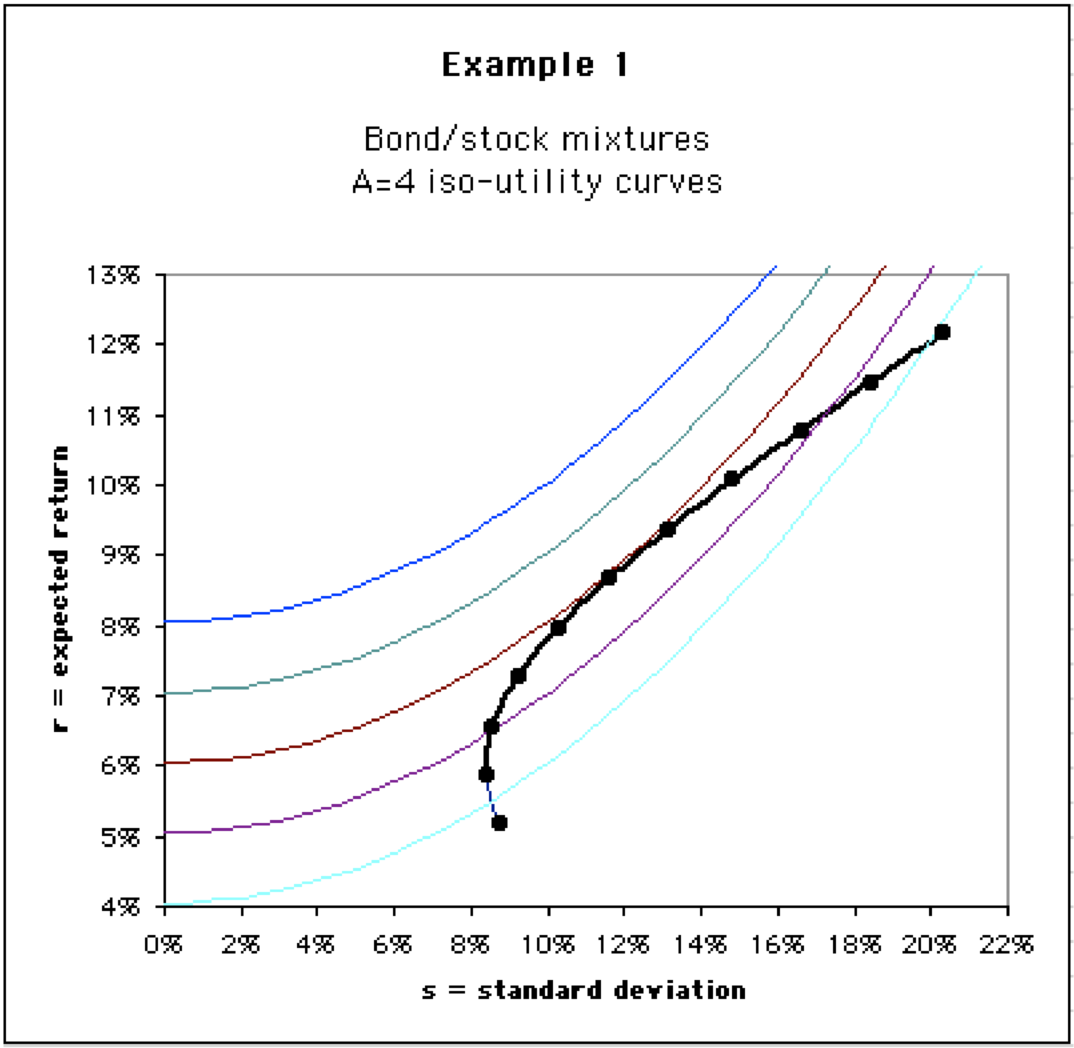

Figure 3: $A = 4$ Iso-Utility Curves

For any constant k, consider the set of all (r, s) pairs for which this function has the value k:

$$\begin{aligned} k &= r - 2s^2 \\ r &= k + 2s^2 \end{aligned}$$

For any fixed value of k, this function plots as a parabola on our graph, as illustrated in Figure 3.

This graph shows five of the parabolas, for k = 4.05%, 5.05%, 6.05%, 7.05%, and 8.05%.

These curves are called *iso-utility curves* or *indifference curves*. For any given curve, the possible investments which plot on the curve all have the same expected utility, and the investor is therefore indifferent among them. The y-intercept of a curve is the value k. This is the investor's certainty equivalent for all the other possible investments on the curve. The higher indifference curves have larger certainty equivalents and larger expected utility, so our investor prefers them to the lower indifference curves.

In the example, consider the bottom curve for $k = 4.05\%$. This curve approximately intersects the 100% stock portfolio on the feasible set curve. Thus our investor is indifferent between a risk-free investment which returns 4.05% and the 100% stock investment which has an expected return of 12.16% and a standard deviation of 20.35%.

Next consider the second-to-bottom curve for $k = 5.05\%$. This indifference curve intersects the feasible set curve at two points, at about 20% bonds and at about 20% stocks. The certainty equivalent 5.05 is higher for this curve than it is for the bottom curve, so the investor prefers any point on the second-to-bottom curve to any point on the bottom curve. In particular, our investor prefers both the 20% bond portfolio and the 20% stock portfolio to the 100% stock portfolio.

Now consider the middle curve for $k = 6.05\%$. This curve is tangent to the feasible set curve at approximately the portfolio which is 45% bonds and 55% stocks. This portfolio is preferred to any of the portfolios on the bottom and second-to-bottom indifference curves.

Now consider the top two curves for $k = 7.05\%$ and $k = 8.05\%$. These curves do not intersect the feasible set curve. While our investor would certainly like to find portfolios on these higher indifference curves, it is unfortunately impossible given the feasible set of investment alternatives under consideration.

The optimal portfolio for our investor is clearly the 45% bond/55% stock portfolio, with a certainty equivalent of 6.05. This portfolio maximizes the investor's expected utility.

Note that the feasible set and the efficient frontier on these kinds of graphs are the same for all investors. Only the indifference curves vary from one investor to another investor and depend on the investor's particular utility function.

To illustrate this point, consider the same example with a negative exponential utility function, only this time change the coefficient of risk aversion A from 4 to 2. Our second investor with $A = 2$ is more risk-tolerant (less risk-averse) than is our first investor with $A = 4$. Figure 4 shows the iso-utility curves for our second investor.

In this example, the feasible set is the same, and the efficient frontier is the same, but the indifference curves are noticeably flatter. The optimal portfolio is

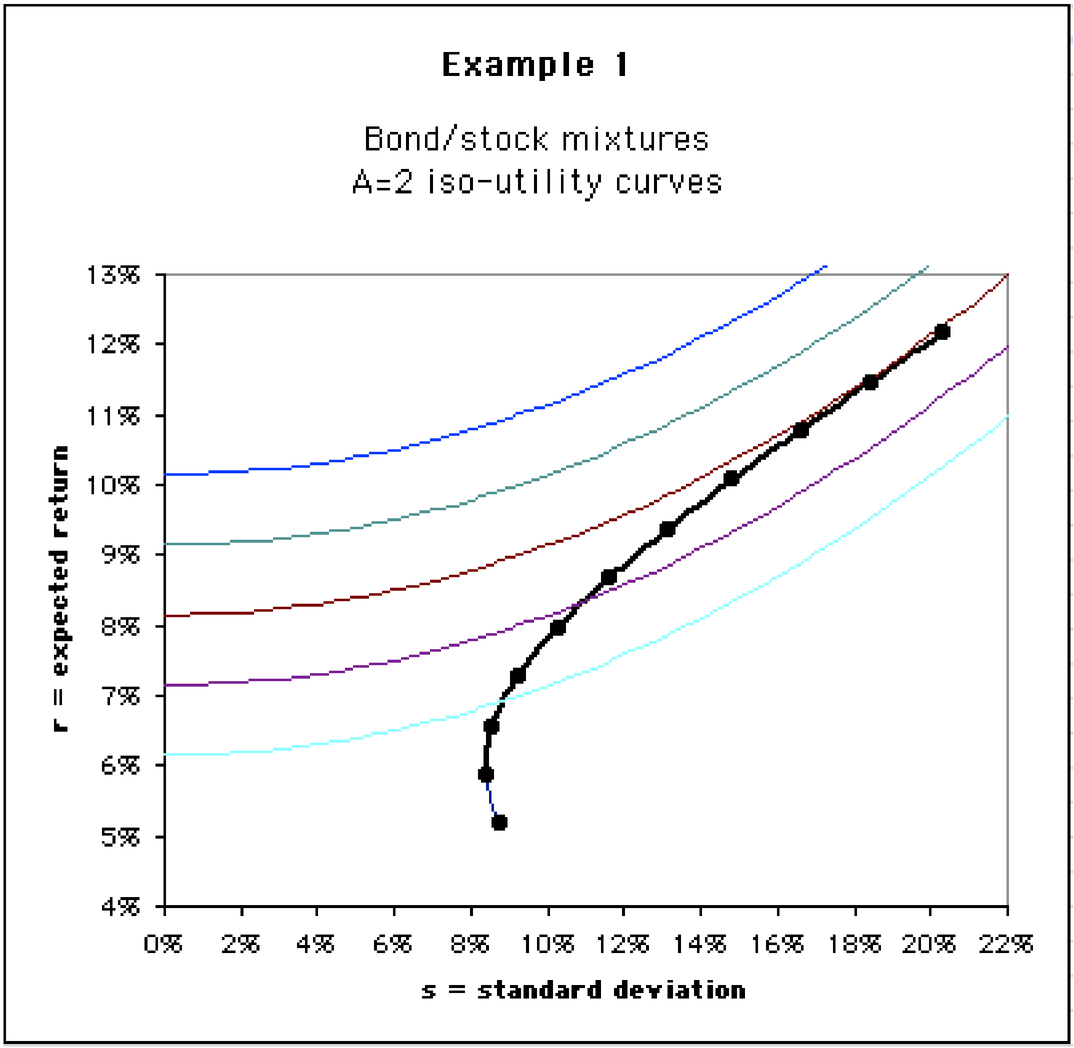

Figure 4: $A = 2$ Iso-Utility Curves

about 90% stocks, with a certainty equivalent of 8.15%. Our second investor is clearly quite a bit more risk-tolerant than is our first investor, and his optimal portfolio is considerably more aggressive.

In general, given a utility function for an individual investor (negative exponential or otherwise), the portfolio optimization problem is to find the indifference curve which is tangent to the efficient frontier. The optimal portfolio for the investor is the one located at the tangency point.

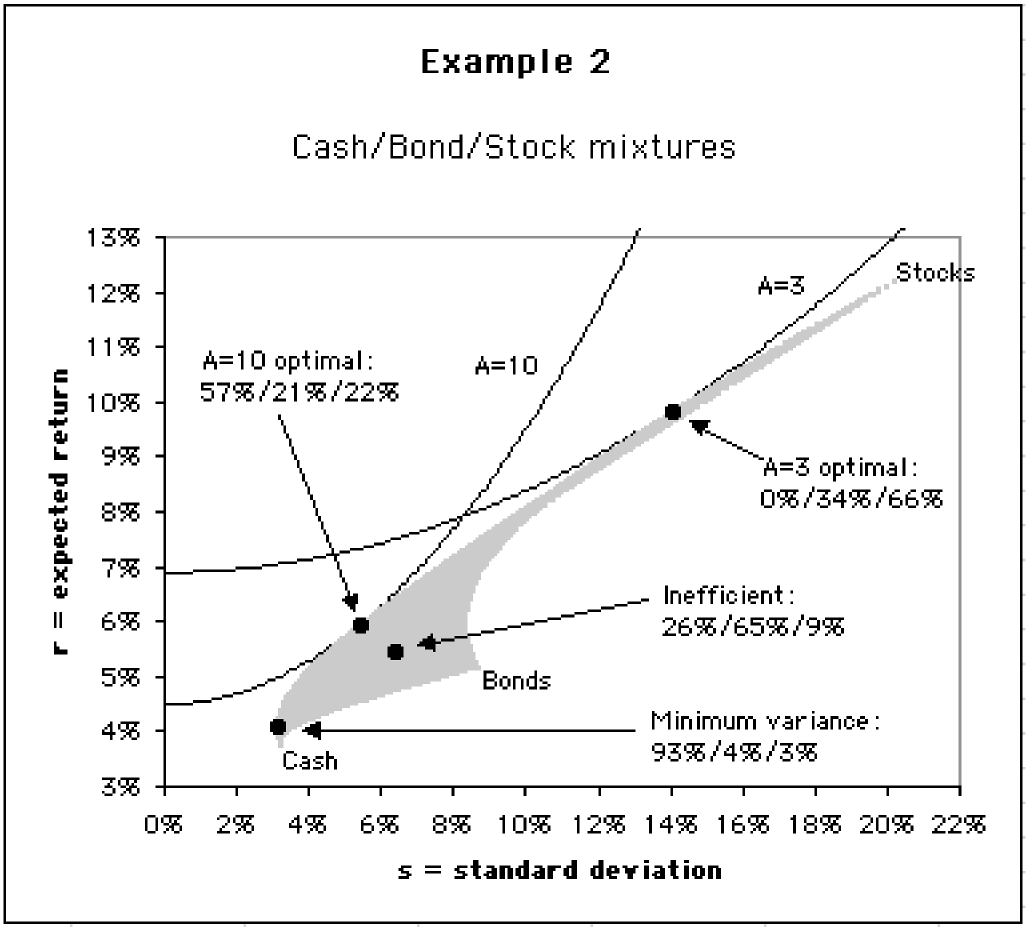

Figure 5: Example 2 – Cash, Bonds and Stocks

3.3.2 Example 2 – Cash, Bonds and Stocks

In our second example we add a third asset to the bonds and stocks of the first example. Our new asset is "cash" (90 day U.S. Treasury bills). Again we use the 1926-1994 data to estimate portfolio expected returns and standard deviations. Figure 5 shows the feasible set in gray. Portfolios combining cash, bonds, and stocks are generated and plotted in increments of 1% in each asset dimension, for a total of 5,151 data points representing portfolios in the feasible set.

Note the typical sail shape of the resulting feasible set when plotted on our risk/return graph. The corners of the sail represent pure portfolios with 100% invested in a single asset. Note that the stock corner has the highest expected return and is efficient, while the cash and bond corners have lower expected return and lower risk and are inefficient. This is also typical. The efficient frontier is the northwest edge of the feasible set ranging from the minimum variance portfolio to the all-stock portfolio.

We have also graphed the indifference curves tangent to the efficient frontier for two investors with negative exponential utility functions. The more risk-averse investor has a coefficient of risk aversion $A = 10$ with an optimal portfolio consisting of 57% cash, 21% bonds, and 22% stocks. The more risk-tolerant investor with $A = 3$ has an optimal portfolio consisting of 0% cash, 34% bonds, and 66% stocks.

4 Lognormal Returns

We now develop the parallel portfolio theory to that presented in section 3, replacing the assumption that returns are normally distributed by the more realistic assumption that they are lognormally distributed, as implied by the random walk model. The results are very similar to those we found in section 3, and the mathematics is only slightly more difficult.

Because lognormal returns are appropriate when modeling long time horizons, we also generalize the presentation to treat time horizon as a parameter.

In section 3, we kept the time horizon t fixed and used simply compounded returns r and their standard deviations s measured over the entire time period.

In this section, we instead permit the time horizon t to vary and we measure return and risk by using instantaneous yearly returns α and their standard deviations σ.

We change our notation from r and s to α and σ to emphasize the fact that they are different measures of return and risk.

4.1 Theorems and Proofs

Lemma 4.1 *Suppose the return on investment I is lognormally distributed under the random walk model with mean continuously compounded yearly return μ and standard deviation of continuously compounded yearly returns σ. Let U be the iso-elastic utility function $U(w) = \frac{w^\lambda - 1}{\lambda}$ where $\lambda < 1$ and $\lambda \neq 0$. Let w_0 be the investor's current wealth. Then the expected utility of wealth $w(t)$ after t years is:*

$$E(U(w(t))) = \frac{1}{\lambda} w_0^\lambda e^{\lambda t(\mu + \frac{1}{2}\lambda\sigma^2)} - \frac{1}{\lambda}$$

Proof:

The ending value $w(t)$ is the lognormally distributed random variable:

$$w(t) = w_0 e^{\sigma\sqrt{t}X + \mu t} \quad \text{where } X \text{ is } N[0,1]$$

$$\begin{aligned}
E(U(w(t))) &= \int_{-\infty}^{\infty} U(w_0 e^{\sigma\sqrt{t}x + \mu t}) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_{-\infty}^{\infty} \frac{(w_0 e^{\sigma\sqrt{t}x + \mu t})^\lambda - 1}{\lambda} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \frac{1}{\lambda} \int_{-\infty}^{\infty} w_0^\lambda e^{\lambda\sigma\sqrt{t}x + \lambda\mu t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - \frac{1}{\lambda} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\lambda}w_0^\lambda e^{\lambda\mu t}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2+\lambda\sigma\sqrt{t}x}dx - \frac{1}{\lambda} \\
&= \text{(complete the square)} \\
&\quad \frac{1}{\lambda}w_0^\lambda e^{\lambda\mu t}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(x-\lambda\sigma\sqrt{t})^2/2+\frac{1}{2}\lambda^2\sigma^2 t}dx - \frac{1}{\lambda} \\
&= \frac{1}{\lambda}w_0^\lambda e^{\lambda\mu t+\frac{1}{2}\lambda^2\sigma^2 t}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(x-\lambda\sigma\sqrt{t})^2/2}dx - \frac{1}{\lambda} \\
&= \text{(substitute } y = x - \lambda\sigma\sqrt{t}) \\
&\quad \frac{1}{\lambda}w_0^\lambda e^{\lambda\mu t+\frac{1}{2}\lambda^2\sigma^2 t}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy - \frac{1}{\lambda} \\
&= \frac{1}{\lambda}w_0^\lambda e^{\lambda\mu t+\frac{1}{2}\lambda^2\sigma^2 t} - \frac{1}{\lambda} \\
&= \frac{1}{\lambda}w_0^\lambda e^{\lambda t(\mu+\frac{1}{2}\lambda\sigma^2)} - \frac{1}{\lambda}
\end{aligned}$$

Lemma 4.2 *Suppose the return on investment I is lognormally distributed under the random walk model with mean continuously compounded yearly return μ and standard deviation of continuously compounded yearly returns σ. Let U be the iso-elastic logarithmic utility function $U(w) = \log(w)$. Let w_0 be the investor's current wealth. Then the expected utility of wealth $w(t)$ after t years is:*

$$E(U(w(t))) = \log(w_0) + \mu t$$

Proof:

The ending value $w(t)$ is the lognormally distributed random variable:

$$w(t) = w_0 e^{\sigma\sqrt{t}X+\mu t} \qquad \text{where } X \text{ is } N[0,1]$$

$$\begin{aligned}
E(U(w(t))) &= \int_{-\infty}^{\infty} U(w_0 e^{\sigma\sqrt{t}x+\mu t})\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \int_{-\infty}^{\infty} \log(w_0 e^{\sigma\sqrt{t}x+\mu t})\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \int_{-\infty}^{\infty} [\log(w_0) + \log(e^{\sigma\sqrt{t}x+\mu t})]\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \log(w_0)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx + \int_{-\infty}^{\infty}(\sigma\sqrt{t}x+\mu t)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \log(w_0) + \sigma\sqrt{t}\int_{-\infty}^{\infty}x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx + \mu t\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \log(w_0) + \sigma\sqrt{t}\times 0 + \mu t \\
&= \log(w_0) + \mu t
\end{aligned}$$

Theorem 4.1 *For time horizon t, a utility-maximizing investor with initial wealth w_0 and an iso-elastic utility function with coefficient of risk aversion A, when faced with a decision among a set of competing feasible investment alternatives F all of whose elements have lognormally distributed returns under the random walk model, acts to select an investment $I \in F$ which maximizes*

$$\alpha_I - \frac{1}{2}A\sigma_I^2$$

where α_I and σ_I are the expected instantaneous yearly return and standard deviation for investment I respectively.

Proof:

Let μ_I be the expected continuously compounded yearly return for investment I. By Theorem 4.1 in [6]:

$$\alpha_I = \mu_I + \frac{1}{2}\sigma_I^2$$

The investor acts to maximize the expected utility of end-of-period wealth.

First consider the case $A \neq 1$. Let $\lambda = 1 - A$. The investor's utility function is $U(w) = \dfrac{w^\lambda - 1}{\lambda}$. By Lemma 4.1, the expected utility of end-of-period wealth is:

$$\begin{aligned} E(U(w(t))) &= \frac{1}{\lambda}w_0^\lambda e^{\lambda t(\mu_I + \frac{1}{2}\lambda\sigma_I^2)} - \frac{1}{\lambda} \\ &= \frac{1}{\lambda}w_0^\lambda e^{\lambda t(\mu_I + \frac{1}{2}\sigma_I^2 - \frac{1}{2}A\sigma_I^2)} - \frac{1}{\lambda} \end{aligned}$$

Maximizing this function is clearly equivalent to maximizing:

$$\mu_I + \frac{1}{2}\sigma_I^2 - \frac{1}{2}A\sigma_I^2 = \alpha_I - \frac{1}{2}A\sigma_I^2$$

(The signs are a bit tricky. Treat the cases $\lambda < 0$ and $\lambda > 0$ separately.)

Now consider the case $A = 1$. The investor's utility function is $U(w) = \log(w)$. By Lemma 4.2, the expected utility of end-of-period wealth is:

$$\begin{aligned} E(U(w(t))) &= \log(w_0) + \mu_I t \\ &= \log(w_0) + (\mu_I + \frac{1}{2}\sigma_I^2 - \frac{1}{2}A\sigma_I^2)t \end{aligned}$$

Once again, maximizing this function is equivalent to maximizing:

$$\mu_I + \frac{1}{2}\sigma_I^2 - \frac{1}{2}A\sigma_I^2 = \alpha_I - \frac{1}{2}A\sigma_I^2$$

Note that the optimal investment in Theorem 4.1 depends only on the investor's coefficient of risk aversion and on the expected return and volatility of the investments in the feasible set. In particular, the optimal investment is independent of both the investor's current wealth and his time horizon.

This is an important result.[6] Recall that the iso-elastic utility functions characterize those investors who have constant relative risk aversion – their relative attitudes towards risk are independent of current wealth. Theorem 4.1 shows that in the random walk model any such investor's relative attitudes towards risk are also independent of time horizon.

Many people find this result counter-intuitive because of the nearly ubiquitous popular opinion that the risk of investing in volatile assets like stocks decreases as an investor's time horizon increases, regardless of the investor's attitudes towards risk and wealth. The result is nonetheless true – if risk attitudes are independent of wealth, they are also necessarily independent of time. The mathematics is inescapable and suggests that perhaps the popular opinion is suspect.

Recall that in the random walk model, for an investment with expected instantaneous yearly return α, expected continuously compounded yearly return μ, and standard deviation of continuously compounded yearly returns σ, we have:

$$\begin{aligned}
e^{\alpha} - 1 &= \text{arithmetic mean yearly return} \\
&= \text{average yearly simply compounded return} \\
&= \text{expected yearly simply compounded return} \\
e^{\mu} - 1 &= e^{\alpha - \frac{1}{2}\sigma^2} - 1 \\
&= \text{geometric mean yearly return} \\
&= \text{median yearly simply compounded return}
\end{aligned}$$

Thus, for a risk-neutral investor with $A = 0$, Theorem 4.1 tells us that the investor acts to maximize the arithmetic mean return. This makes sense, because such an investor is totally oblivious to risk. Diversification means nothing to such an investor, who given an asset allocation problem invariably chooses to invest 100% of his wealth in the single asset with the highest expected return, no matter how risky that asset might be.

For an investor with $A = 1$ and a logarithmic utility function, Theorem 4.1 tells us that the investor acts to maximize the geometric mean return. Such investors are very risk-tolerant and typically have optimal portfolios containing very large percentages of risky assets. Most investors are more risk-averse and have coefficients of risk aversion significantly greater than 1.

[6] In this paper we have made the assumption that the investor makes a single investment decision at the beginning of the time period and does not change his decision during the time period. It turns out that for iso-elastic utility the same result holds even if the investor is permitted to change his investment decision continuously during the time period. See Section 4.4 in Merton [3].

Theorem 4.2 *Suppose I_1 and I_2 are two investments with lognormally distributed returns, with expected instantaneous yearly returns α_1 and α_2 and standard deviations of instantaneous yearly returns σ_1 and σ_2 respectively, and with initial wealth w_0 for both investments. Then I_1 is more efficient than I_2 if and only if:*

$$\alpha_1 \geq \alpha_2 \text{ and } \sigma_1 \leq \sigma_2$$

with strict inequality holding in at least one of the inequalities. This result holds for all time horizons t.

Proof:

For an investment I with lognormally distributed return, with expected continuously compounded yearly return μ and standard deviation of continuously compounded yearly returns σ, with initial wealth w_0, the ending value $w(t)$ at time t is the lognormally distributed random variable:

$$\begin{aligned} w(t) &= w_0 e^{\sigma\sqrt{t}X+\mu t} \quad \text{where } X \text{ is } N[0,1] \\ &= w_0 e^{\sigma\sqrt{t}X+(\alpha-\frac{1}{2}\sigma^2)t} \quad \text{where } \alpha = \mu + \tfrac{1}{2}\sigma^2 \end{aligned}$$

Let U be any utility function and define:

$$u(\alpha,\sigma) = E(U(w(t))) = \int_{-\infty}^{\infty} U(w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t})\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

We prove the "if" direction by showing that $\frac{\partial\mu}{\partial\alpha} > 0$ and $\frac{\partial\mu}{\partial\sigma} < 0$ for $\sigma > 0$.

$$\begin{aligned} \frac{\partial\mu}{\partial\alpha} &= \int_{-\infty}^{\infty} U'(w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t}) w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t} t \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &> 0 \quad \text{because } U' > 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial\mu}{\partial\sigma} &= \int_{-\infty}^{\infty} U'(w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t}) w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t}(\sqrt{t}x-\sigma t)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= w_0\sqrt{t}e^{(\alpha-\frac{1}{2}\sigma^2)t}\int_{-\infty}^{\infty} U'(w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t})(x-\sigma\sqrt{t})\frac{1}{\sqrt{2\pi}}e^{-x^2/2+\sigma\sqrt{t}x}dx \\ &= w_0\sqrt{t}e^{(\alpha-\frac{1}{2}\sigma^2)t}\int_{-\infty}^{\infty} U'(w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t})(x-\sigma\sqrt{t})\frac{1}{\sqrt{2\pi}}e^{-(x-\sigma\sqrt{t})^2/2+\frac{1}{2}\sigma^2 t}dx \\ &= w_0\sqrt{t}e^{\alpha}\int_{-\infty}^{\infty} U'(w_0 e^{\sigma\sqrt{t}x+(\alpha-\frac{1}{2}\sigma^2)t})(x-\sigma\sqrt{t})\frac{1}{\sqrt{2\pi}}e^{-(x-\sigma\sqrt{t})^2/2}dx \\ &= (\text{substitute } y = x-\sigma\sqrt{t}) \\ &\quad w_0\sqrt{t}e^{\alpha}\int_{-\infty}^{\infty} U'(w_0 e^{\sigma\sqrt{t}y+(\alpha+\frac{1}{2}\sigma^2)t})y\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\ &= w_0\sqrt{t}e^{\alpha}\left[\int_{-\infty}^{0} U'(w_0 e^{\sigma\sqrt{t}y+(\alpha+\frac{1}{2}\sigma^2)t})y\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy + \right. \end{aligned}$$

$$\int_0^\infty U'(w_0 e^{\sigma\sqrt{t}y+(\alpha+\frac{1}{2}\sigma^2)t}) y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \Big]$$

$$= w_0\sqrt{t}e^\alpha \Big[\int_0^\infty U'(w_0 e^{\sigma\sqrt{t}(-y)+(\alpha+\frac{1}{2}\sigma^2)t})(-y) \frac{1}{\sqrt{2\pi}} e^{-(-y)^2/2} dy +$$

$$\int_0^\infty U'(w_0 e^{\sigma\sqrt{t}y+(\alpha+\frac{1}{2}\sigma^2)t}) y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \Big]$$

$$= w_0\sqrt{t}e^\alpha \int_0^\infty \Big[U'(w_0 e^{\sigma\sqrt{t}y+(\alpha+\frac{1}{2}\sigma^2)t}) - U'(w_0 e^{-\sigma\sqrt{t}y+(\alpha+\frac{1}{2}\sigma^2)t}) \Big] y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$U'' < 0$, so U' is a decreasing function, so whenever $a > b$ we must have $U'(a) < U'(b)$ and hence $U'(a) - U'(b) < 0$. Thus, when $y > 0$ and $\sigma > 0$, the expression in square brackets above is negative, and hence the entire integral is negative and we have our result.

To prove the "only if" direction it suffices to show that if $\alpha_1 < \alpha_2$ and $\sigma_1 < \sigma_2$ then there is at least one utility function U with $E(U(w_1(t))) < E(U(w_2(t)))$ and at least one other utility function V with $E(V(w_1(t))) > E(V(w_2(t)))$.

We use the iso-elastic utility functions and Theorem 4.1 in this half of the proof.

By Theorem 4.1, our investor acts to maximize $\alpha - \frac{1}{2}A\sigma^2$. In order for the second investment to have greater expected utility than the first investment we must have:

$$\alpha_2 - \frac{1}{2}A\sigma_2^2 > \alpha_1 - \frac{1}{2}A\sigma_1^2$$

Solve this inequality for A:

$$A < 2\frac{\alpha_2 - \alpha_1}{\sigma_2^2 - \sigma_1^2} > 0$$

Thus, for sufficiently small values of A, the expected utility of the second investment is greater than the expected utility of the first investment. For large values of A the situation is reversed, and the investor prefers the first investment to the second investment. Thus neither investment is more efficient than the other one.

4.2 Risk/Return Graphs

Risk/return graphs in the lognormal model are very similar in appearance to the ones in the normal model. The only difference is that we graph σ on the x axis and $\alpha = \mu + \frac{1}{2}\sigma^2$ on the y axis.

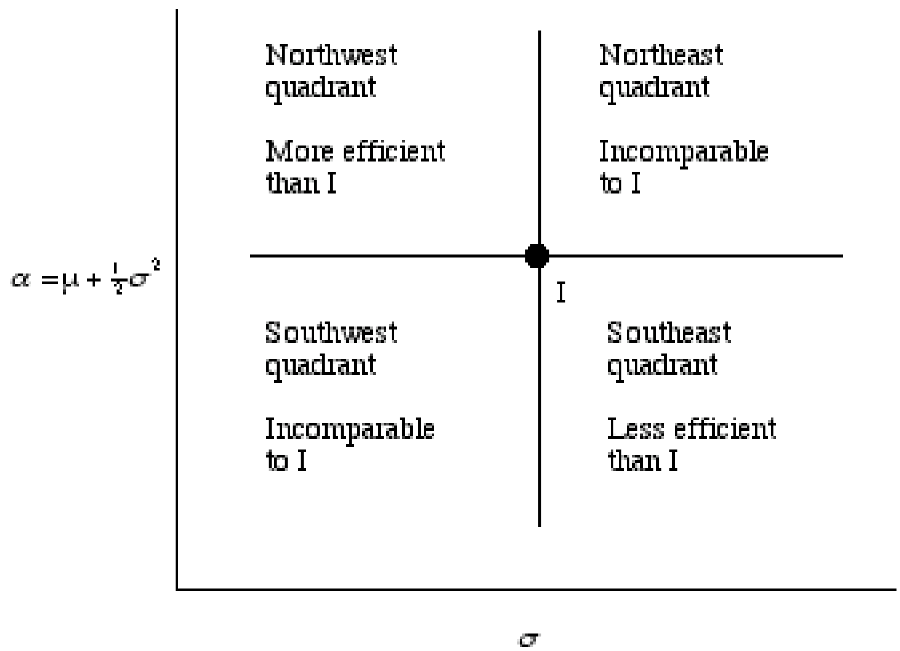

Figure 6: Risk/Return Graphs

4.3 Examples

We now redo the examples in section 3.3 using the lognormal model instead of the normal model. In the graphs, on the x axis, labeled "Risk," we plot $\sigma =$ the standard deviation of the continuously compounded yearly returns. On the y axis, labeled "Return," we plot $\alpha = \mu + \frac{1}{2}\sigma^2$, where $\mu =$ the mean continuously compounded yearly return. As in section 3.3, we estimate the parameters μ and σ using historical market return data from 1926-1994.

4.3.1 Example 1 – Bonds and Stocks

Figure 7 shows the risk/return graph for mixtures of 20 year U.S. Treasury bonds and S&P 500 large U.S. stocks.

Note that the minimum variance portfolio is the same as in section 3.3.1, to the nearest whole percentage.

In Figure 3 we graphed iso-utility curves for the negative exponential utility function with coefficient of risk-aversion $A = 4$. We measured utility as a function of return rather than end-of-period wealth.

In Figure 8 we instead graph iso-utility curves for the iso-elastic utility function with coefficient of risk-aversion $A = 4$. In this case, in the lognormal model, there is no need to use the "hack" of measuring utility as a function of return rather than end-of-period wealth. Our iso-elastic utility function is the regular one that measures utility as a function of end-of-period wealth. Note that the optimal portfolio at the tangency point of the middle indifference curve is approximately the same, about 45% bonds and 55% stocks.

In Figure 9 we graph iso-utility curves for the iso-elastic utility function with coefficient of risk-aversion $A = 2$. Changing the coefficient of risk-aversion from 4 to 2 once again gives very similar results in the lognormal model as it gave in the normal model. The optimal portfolio is once again quite aggressive and approximately the same, about 90% stocks and 10% bonds.

4.3.2 Example 2 – Cash, Bonds and Stocks

Figure 10 shows the risk/return graph for mixtures of cash, bonds, and stocks.

Once again, in our final three asset example, we see that the results for the lognormal model are very similar to the ones we found for the normal model.

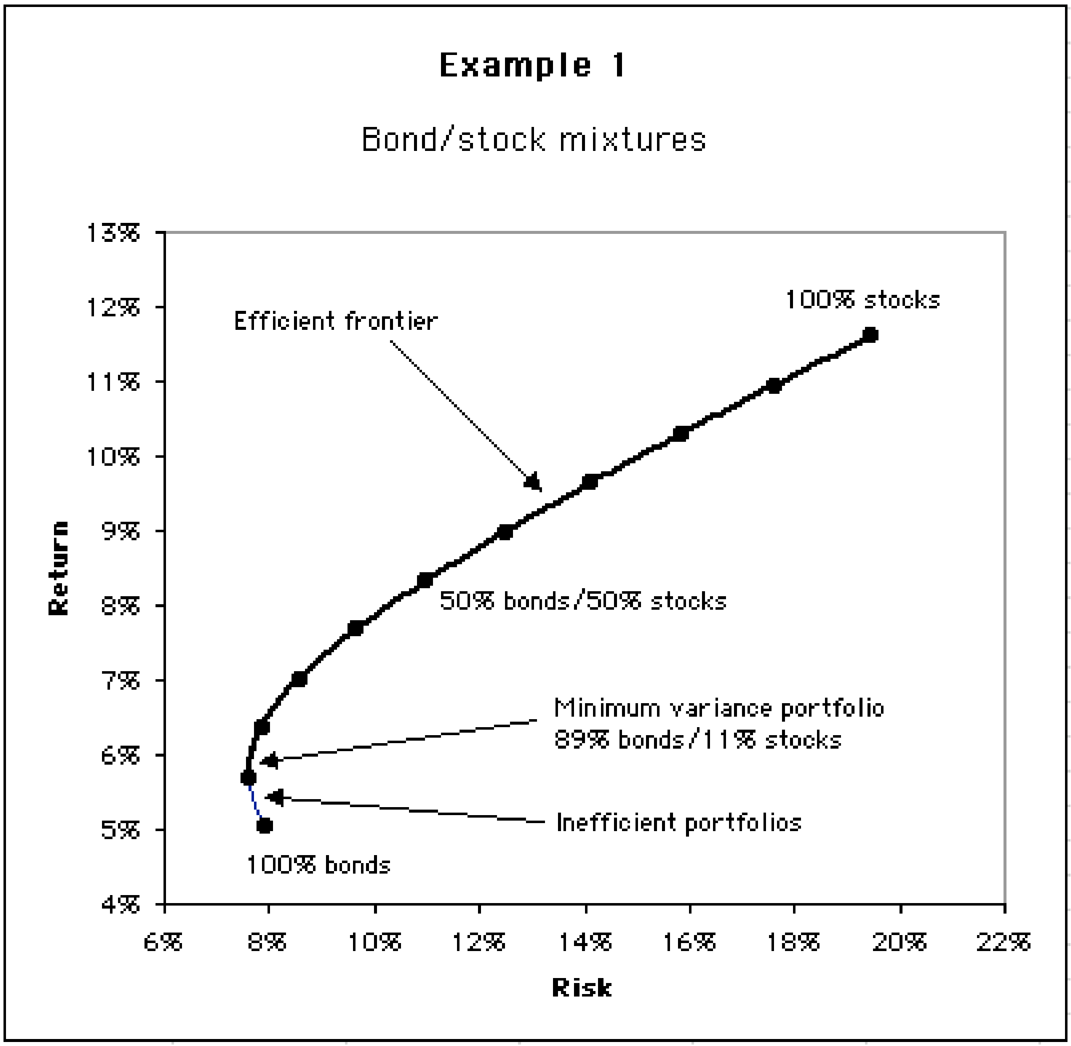

Figure 7: Example 1 – Bonds and Stocks

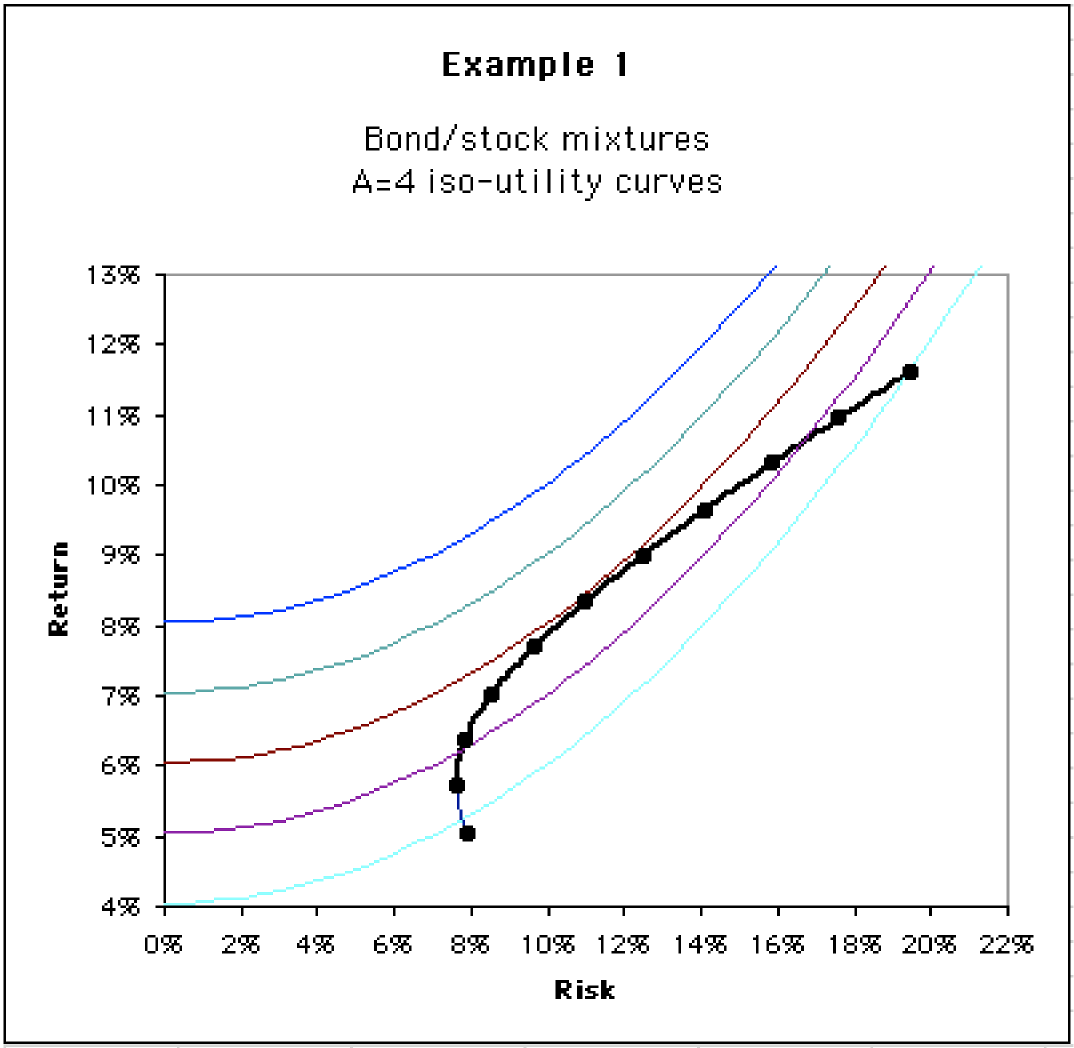

Figure 8: $A = 4$ Iso-Utility Curves

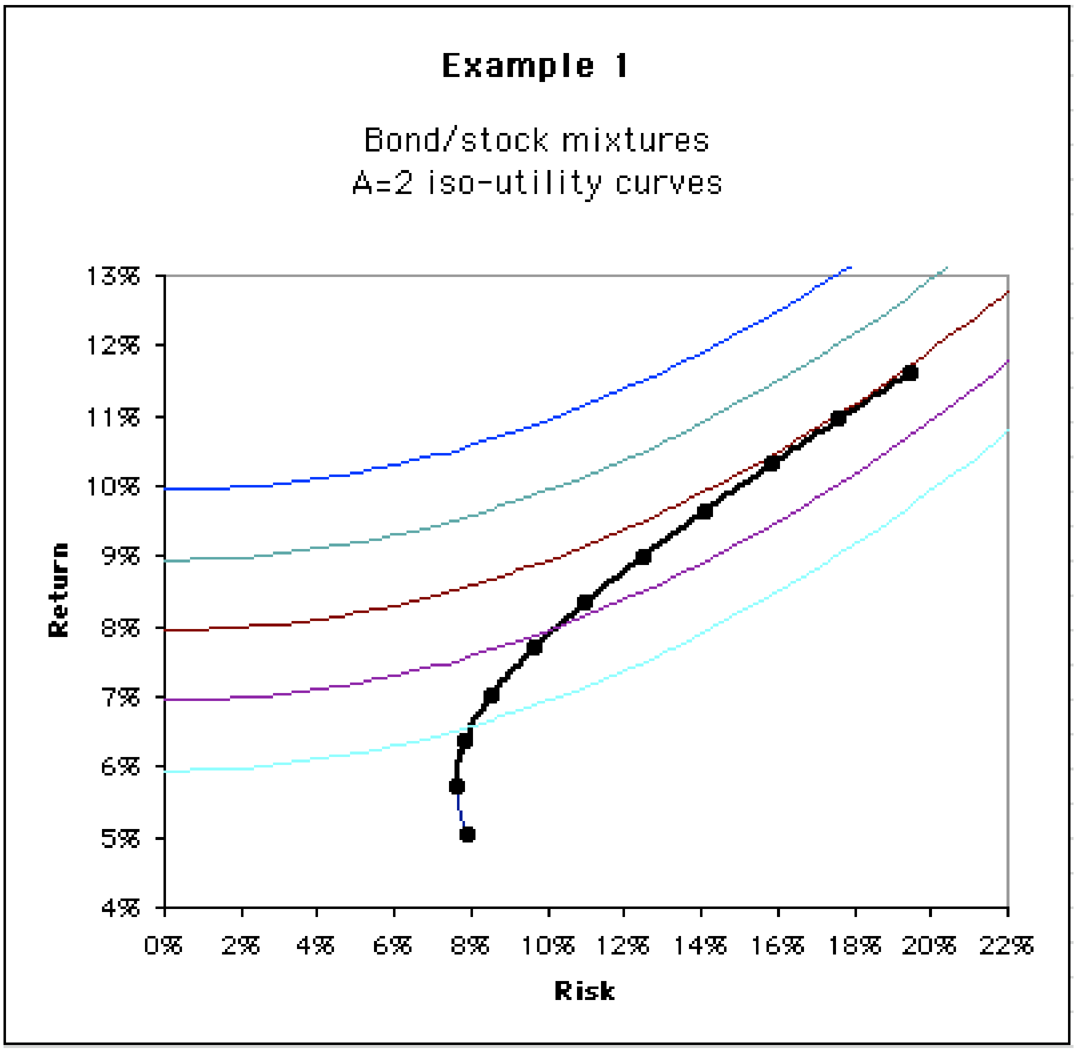

Figure 9: $A = 2$ Iso-Utility Curves

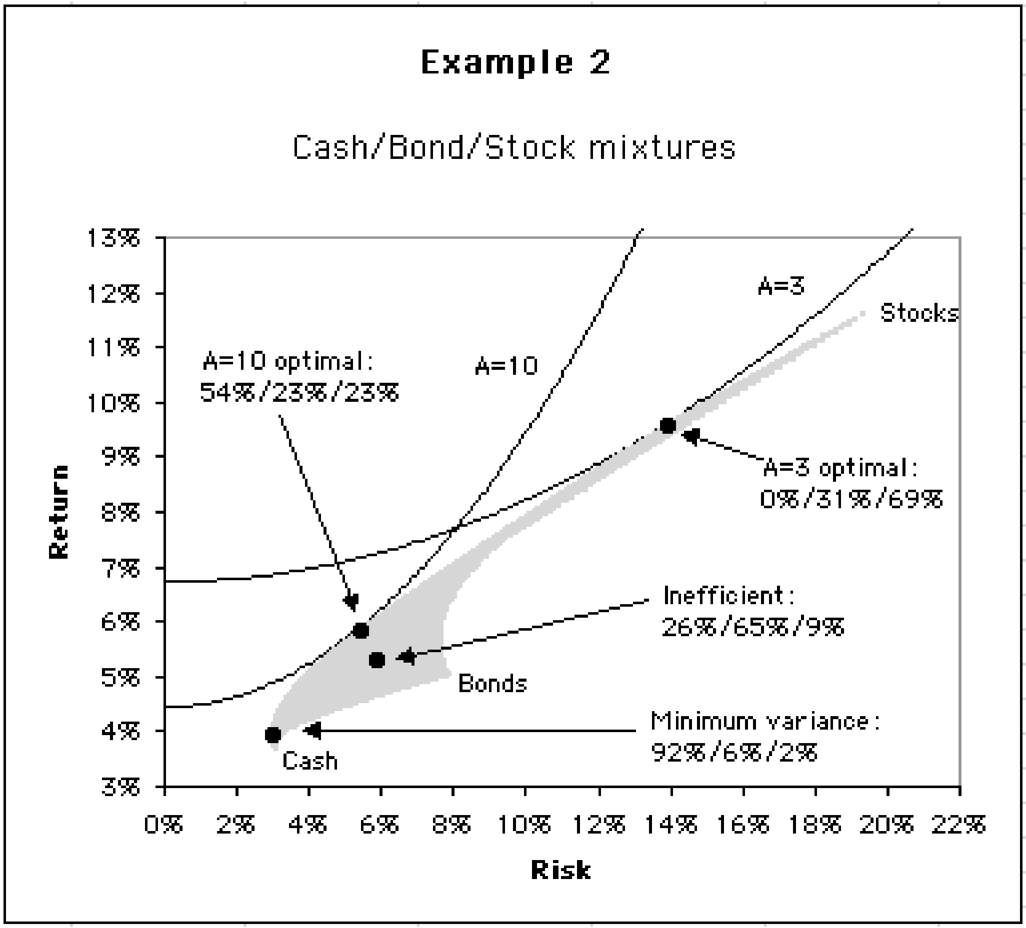

Figure 10: Example 2 – Cash, Bonds and Stocks

References

[1] Zvi Bodie and Robert C. Merton. *Finance*. Prentice-Hall, preliminary edition, 1998.

[2] Harry M. Markowitz. *Portfolio Selection: Efficient Diversification of Investments*. Blackwell, second edition, 1991. (Originally published in 1959.).

[3] Robert C. Merton. *Continuous-Time Finance*. Blackwell, 1990.

[4] John Norstad. An introduction to utility theory. http://www.norstad.org/finance, Mar 1999.

[5] John Norstad. The normal and lognormal distributions. http://www.norstad.org/finance, Feb 1999.

[6] John Norstad. Random walks. http://www.norstad.org/finance, Jan 2005.

[7] William F. Sharpe. Macro-investment analysis (an electronic work-in-progress). http://www.stanford.edu/~wfsharpe/mia/mia.htm.