The Normal and Lognormal Distributions

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Abstract

The basic properties of the normal and lognormal distributions, with full proofs.

We assume familiarity with elementary probability theory and with college-level calculus.

1 Definitions and Summary of the Propositions

Proposition 1: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = 1$

Proposition 2: $\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \mu$

Proposition 3: $\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \mu^2 + \sigma^2$

Definition 1 *The* normal distribution $N[\mu, \sigma^2]$ *is the probability distribution defined by the following density function:*

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Note that Proposition 1 verifies that this is a valid density function (its integral from $-\infty$ to ∞ is 1).

Definition 2 *The* lognormal distribution $LN[\mu, \sigma^2]$ *is the distribution of* e^X *where* X *is* $N[\mu, \sigma^2]$.

Proposition 4: If X is $N[\mu, \sigma^2]$ then $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$.

Proposition 5: If Y is $LN[\mu, \sigma^2]$ then $E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$ and $\text{Var}(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$.

Proposition 6: If X is $N[\mu, \sigma^2]$ then $aX + b$ is $N[a\mu + b, a^2\sigma^2]$.

Proposition 7: If X is $N[\mu_1, \sigma_1^2]$, Y is $N[\mu_2, \sigma_2^2]$, and X and Y are independent, then $X + Y$ is $N[\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2]$.

Corollary 1: If X_i are independent $N[\mu, \sigma^2]$ for $i = 1 \ldots n$ then $\sum_{i=1}^{n} X_i$ is $N[n\mu, n\sigma^2]$.

Corollary 2: If Y_i are independent $LN[\mu, \sigma^2]$ for $i = 1 \ldots n$ then $\prod_{i=1}^{n} Y_i$ is $LN[n\mu, n\sigma^2]$.

Proposition 8: The probability density function of $LN[\mu, \sigma^2]$ is:

$$\frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log(x)-\mu)^2/2\sigma^2}$$

2 Proofs of the Propositions

Proposition 1

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = 1$$

Proof:

First assume that $\mu = 0$ and $\sigma = 1$. Let:

$$a = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Then:

$$\begin{aligned} a^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dx dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy \end{aligned}$$

Apply the polar transformation $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$:

$$\begin{aligned} a^2 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r dr d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[-e^{-r^2/2}\right]_0^{\infty} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} [0 - (-1)] d\theta \\ &= \frac{1}{2\pi} 2\pi \\ &= 1 \end{aligned}$$

$a > 0$, and we just showed that $a^2 = 1$, so we must have $a = 1$.

For the general case, apply the transformation $y = \dfrac{x-\mu}{\sigma}$, $dy = \dfrac{dx}{\sigma}$:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2} \sigma dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= 1 \end{aligned}$$

Proposition 2

$$\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \mu$$

Proof:

First assume that $\mu = 0$ and $\sigma = 1$.

$$\begin{aligned}
\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} x e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \left[-e^{-x^2/2}\right]_{-n}^{n} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} [(-e^{-n^2/2}) - (-e^{-n^2/2})] \\
&= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} 0 \\
&= 0
\end{aligned}$$

For the general case, apply the transformation $y = \frac{x-\mu}{\sigma}$, $dy = \frac{dx}{\sigma}$:

$$\begin{aligned}
\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx &= \int_{-\infty}^{\infty} (\mu + \sigma y) \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2} \sigma dy \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \sigma \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \mu \times 1 + \sigma \times 0 \qquad \text{(by Proposition 1)} \\
&= \mu
\end{aligned}$$

Proposition 3

$$\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \mu^2 + \sigma^2$$

Proof:

First assume that $\mu = 0$ and $\sigma = 1$.

Integrate by parts using $f = x$, $f' = 1$, $g = -e^{-x^2/2}$, $g' = xe^{-x^2/2}$:

$$\begin{aligned}
\int_{-n}^{n} x^2 e^{-x^2/2} dx &= \int_{-n}^{n} f g' dx \\
&= f(n)g(n) - f(-n)g(-n) - \int_{-n}^{n} f' g dx \\
&= (-ne^{-n^2/2}) - (ne^{-n^2/2}) - \int_{-n}^{n} -e^{-x^2/2} dx
\end{aligned}$$

$$
\begin{aligned}
&= -2ne^{-n^2/2} - \int_{-n}^{n} -e^{-x^2/2}dx \\
&= \int_{-n}^{n} e^{-x^2/2}dx - 2ne^{-n^2/2}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx &= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} x^2 e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \left[\int_{-n}^{n} e^{-x^2/2}dx - 2ne^{-n^2/2}\right] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx - \lim_{n\to\infty} 2ne^{-n^2/2} \\
&= 1 - \lim_{n\to\infty} 2ne^{-n^2/2} \qquad \text{(by Proposition 1)}
\end{aligned}
$$

All that remains is to show that the last limit above is 0. We do this using L'Hôpital's rule:

$$
\lim_{n\to\infty} 2ne^{-n^2/2} = \lim_{n\to\infty} \frac{2n}{e^{n^2/2}} = \lim_{n\to\infty} \frac{2}{ne^{n^2/2}} = 0
$$

For the general case, apply the transformation $y = \dfrac{x-\mu}{\sigma}$, $dy = \dfrac{dx}{\sigma}$:

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}dx &= \int_{-\infty}^{\infty} (\mu+\sigma y)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2}\sigma dy \\
&= \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy + \\
&\quad 2\mu\sigma \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy + \\
&\quad \sigma^2 \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy \\
&= \mu^2 \times 1 + 2\mu\sigma \times 0 + \sigma^2 \times 1 \\
&\qquad \text{(by Propositions 1 and 2)} \\
&= \mu^2 + \sigma^2
\end{aligned}
$$

Proposition 4 *If X is $N[\mu,\sigma^2]$ then $E(X)=\mu$ and $Var(X)=\sigma^2$.*

Proof:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}dx = \mu \quad \text{(by Proposition 2)}\\
\mathrm{Var}(X) &= \mathrm{E}(X^2)-\mathrm{E}(X)^2 \quad \text{(by Proposition 1 in reference [1])}\\
&= \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}dx - \mu^2\\
&= \mu^2+\sigma^2-\mu^2 \quad \text{(by Proposition 3)}\\
&= \sigma^2
\end{aligned}$$

Proposition 5 *If Y is $LN[\mu,\sigma^2]$ then $E(Y)=e^{\mu+\frac{1}{2}\sigma^2}$ and $Var(Y)=e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$.*

Proof:

$Y=e^X$ where X is $N[\mu,\sigma^2]$. First assume that $\mu=0$:

$$\begin{aligned}
E(Y) &= E(e^X) = \int_{-\infty}^{\infty} e^x\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}dx\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{2x\sigma^2-x^2}{2\sigma^2}}dx\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\sigma^2)^2+\sigma^4}{2\sigma^2}}dx\\
&= e^{\frac{1}{2}\sigma^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\sigma^2)^2/2\sigma^2}dx\\
&= e^{\frac{1}{2}\sigma^2}\times 1 \quad \text{(by Proposition 1)}\\
&= e^{\frac{1}{2}\sigma^2}\\
E(Y^2) &= E(e^{2X}) = \int_{-\infty}^{\infty} e^{2x}\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}dx\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{4x\sigma^2-x^2}{2\sigma^2}}dx\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-2\sigma^2)^2+4\sigma^4}{2\sigma^2}}dx\\
&= e^{2\sigma^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-2\sigma^2)^2/2\sigma^2}dx\\
&= e^{2\sigma^2}\times 1 \quad \text{(by Proposition 1)}\\
&= e^{2\sigma^2}\\
\mathrm{Var}(Y) &= \mathrm{E}(Y^2)-\mathrm{E}(Y)^2 \quad \text{(by Proposition 1 in reference [1])}\\
&= e^{2\sigma^2}-(e^{\frac{1}{2}\sigma^2})^2 = e^{2\sigma^2}-e^{\sigma^2} = e^{\sigma^2}(e^{\sigma^2}-1)
\end{aligned}$$

For the general case, apply the transformation $y = x - \mu$, $dy = dx$:

$$\begin{aligned}
E(Y) &= E(e^X) = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \int_{-\infty}^{\infty} e^{\mu+y} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= e^{\mu} \int_{-\infty}^{\infty} e^{y} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= \text{(by the same reasoning as for the case } \mu = 0 \text{ above)} \\
&\quad e^{\mu} e^{\frac{1}{2}\sigma^2} \\
&= e^{\mu+\frac{1}{2}\sigma^2} \\
E(Y^2) &= E(e^{2X}) = \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \int_{-\infty}^{\infty} e^{2(y+\mu)} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= e^{2\mu} \int_{-\infty}^{\infty} e^{2y} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= \text{(by the same reasoning as for the case } \mu = 0 \text{ above)} \\
&\quad e^{2\mu} e^{2\sigma^2} \\
&= e^{2\mu+2\sigma^2} \\
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 \quad \text{(by Proposition 1 in reference [1])} \\
&= e^{2\mu+2\sigma^2} - (e^{\mu+\frac{1}{2}\sigma^2})^2 \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\
&= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)
\end{aligned}$$

Proposition 6 *If X is $N[\mu, \sigma^2]$ then $aX + b$ is $N[a\mu + b, a^2\sigma^2]$.*

Proof:

$$\begin{aligned}
\mathrm{Prob}(aX + b < k) &= \mathrm{Prob}(X < (k-b)/a) \\
&= \int_{-\infty}^{(k-b)/a} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \text{(apply the transformation } y = ax + b,\ dy = adx) \\
&\quad \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{y-b}{a}-\mu\right)^2/2\sigma^2} \frac{1}{a} dy \\
&= \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}(a\sigma)} e^{-(y-(a\mu+b))^2/2a^2\sigma^2} dy
\end{aligned}$$

The last term above is the cumulative density function for $N[a\mu + b, a^2\sigma^2]$, so we have our result.

Proposition 7 *If X is $N[\mu_1, \sigma_1^2]$, Y is $N[\mu_2, \sigma_2^2]$, and X and Y are independent, then $X + Y$ is $N[\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2]$.*

Proof:

First assume that X is $N[0, 1]$ and Y is $N[0, \sigma^2]$. Then:

$$
\begin{aligned}
\text{Prob}(X + Y < k) &= \int\int_{x+u<k} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}\sigma} e^{-u^2/2\sigma^2} dx du \\
&= \text{(apply the transformation } u = \sigma y) \\
&\quad \frac{1}{2\pi\sigma} \int\int_{x+\sigma y<k} e^{-(x^2+y^2)/2} \sigma dx dy \\
&= \frac{1}{2\pi} \int\int_{x+\sigma y<k} e^{-(x^2+y^2)/2} dx dy \qquad (1)
\end{aligned}
$$

At this point we temporarily make the assumption that $k \geq 0$. Figure 1 shows the area over which we are integrating. It is the half of the plane below and to the left of the line $x + \sigma y = k$.

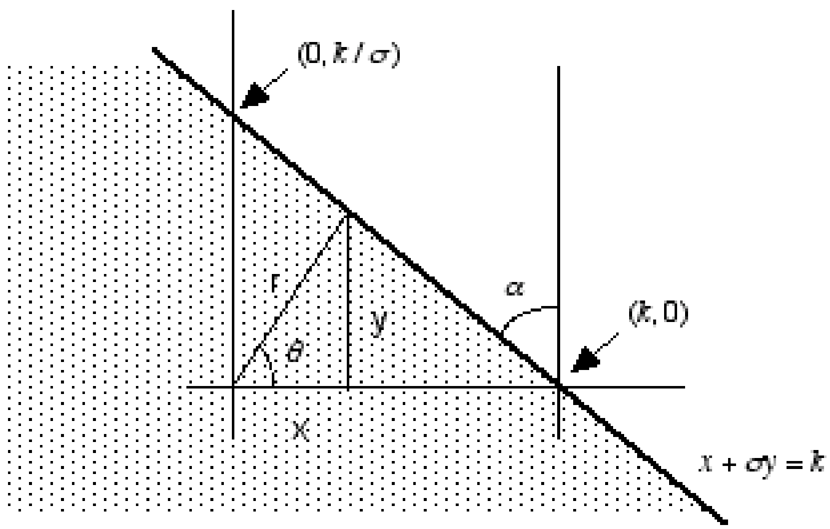

Figure 1: Area of Integration

Note the following relationships:

$$x = r\cos\theta \qquad r = \frac{k}{\cos\theta + \sigma\sin\theta}$$

$$y = r\sin\theta \qquad \tan\alpha = k/(k/\sigma) = \sigma$$

$$r\cos\theta + \sigma r\sin\theta = x + \sigma y = k \qquad \alpha = \arctan\sigma$$

We'll use the following technique to prove the result. First we'll convert the double integral above to polar coordinates. Then we'll rotate the result by $-\alpha$ so that the graph above becomes the one shown in 2 below. Then we'll show that the resulting integral is the same as the one for $\text{Prob}(Z < k)$ where Z is $N[0, 1+\sigma^2]$.

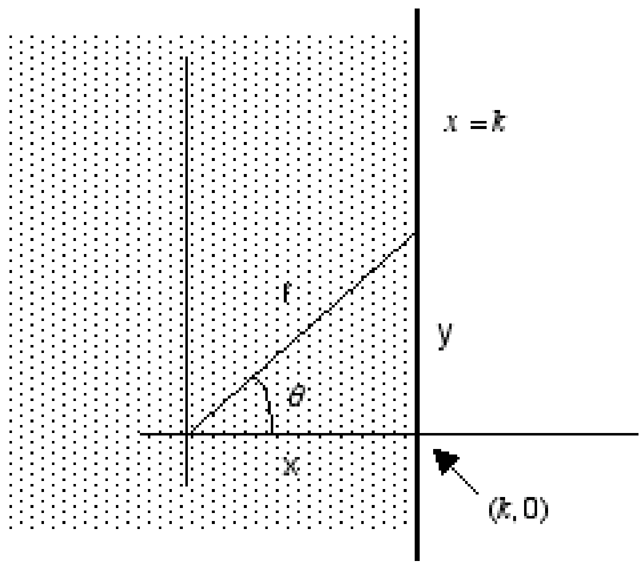

Figure 2: Area of Integration Rotated

Note that $r\cos\theta = x = k$, and $r = k/cos\theta$.

Apply the polar transformation $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$ to equation (1):

$$\begin{aligned}
\text{Prob}(X+Y<k) &= \frac{1}{2\pi}\int\int_{x+\sigma y<k} e^{-(x^2+y^2)/2}dxdy \\
&= \frac{1}{2\pi}\int\int_{x+\sigma y<k} re^{-r^2/2}drd\theta \\
&= \frac{1}{2\pi}\int_{\pi/2+\alpha}^{3\pi/2+\alpha}\int_0^{\infty} re^{-r^2/2}drd\theta + \\
&\quad \frac{1}{2\pi}\int_{-\pi/2+\alpha}^{3\pi/2+\alpha}\int_0^{\frac{k}{\cos\theta+\sigma\sin\theta}} re^{-r^2/2}drd\theta \qquad (2)
\end{aligned}$$

Note that we have made use of the assumption that $k \geq 0$ at this point to split the area over which we are integrating into two regions. In the first region, θ varies from $\pi/2+\alpha$ to $3\pi/2+\alpha$, and the vector at the origin with angle θ does not intersect the line $x+\sigma y$, so r varies from 0 to ∞. In the second region, θ

varies from $-\pi/2+\alpha$ to $\pi/2+\alpha$, and the vector does intersect the line, so r varies from 0 to $k/(\cos\theta+\sigma\sin\theta)$.

We want to apply the transformation $\lambda=\theta-\alpha$ to rotate. We first must calculate what happens to the upper limit of integration in the last integral above under this transformation.

$$\begin{aligned}\frac{k}{\cos\theta+\sigma\sin\theta} &= \frac{k}{\cos(\lambda+\alpha)+\sigma\sin(\lambda+\alpha)}\\ &= \frac{k}{\cos(\lambda+\arctan\sigma)+\sigma\sin(\lambda+\arctan\sigma)} \qquad (3)\end{aligned}$$

We now apply some trigonometric identities:

$$\begin{aligned}\cos(\lambda+\arctan\sigma) &= \cos\lambda\cos(\arctan\sigma)-\sin\lambda\sin(\arctan\sigma)\\ \sin(\lambda+\arctan\sigma) &= \sin\lambda\cos(\arctan\sigma)+\cos\lambda\sin(\arctan\sigma)\\ \cos(\arctan\sigma) &= \frac{1}{\sqrt{1+\sigma^2}}\\ \sin(\arctan\sigma) &= \frac{\sigma}{\sqrt{1+\sigma^2}}\\ \cos(\lambda+\arctan\sigma) &= \frac{\cos\lambda-\sigma\sin\lambda}{\sqrt{1+\sigma^2}} \qquad (4)\\ \sigma\sin(\lambda+\arctan\sigma) &= \frac{\sigma\sin\lambda+\sigma^2\cos\lambda}{\sqrt{1+\sigma^2}} \qquad (5)\end{aligned}$$

Adding equations (4) and (5) gives:

$$\begin{aligned}\cos(\lambda+\arctan\sigma)+\sigma\sin(\lambda+\arctan\sigma) &= \frac{\cos\lambda+\sigma^2\cos\lambda}{\sqrt{1+\sigma^2}}\\ &= \frac{\cos\lambda(1+\sigma^2)}{\sqrt{1+\sigma^2}}\\ &= \cos\lambda\sqrt{1+\sigma^2} \qquad (6)\end{aligned}$$

We can now do our rotation under the transformation $\lambda=\theta-\alpha$. Equations (2), (3) and (6) give:

$$\begin{aligned}\mathrm{Prob}(X+Y<k) &= \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\int_0^{\infty} re^{-r^2/2}drd\lambda+\\ &\quad \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\int_0^{\frac{k}{\cos\lambda\sqrt{1+\sigma^2}}} re^{-r^2/2}drd\lambda\\ &= \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\left[-e^{-r^2/2}\right]_0^{\infty}d\lambda+\\ &\quad \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left[-e^{-r^2/2}\right]_0^{\frac{k}{\cos\lambda\sqrt{1+\sigma^2}}}d\lambda\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}[1]\,d\lambda + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left[-e^{\frac{-k^2}{2\cos^2\lambda(1+\sigma^2)}} - (-1)\right]d\lambda \\
&= 1 - \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{\frac{-k^2}{2\cos^2\lambda(1+\sigma^2)}}\,d\lambda \qquad (7)
\end{aligned}
$$

Now we turn our attention to evaluating $\text{Prob}(Z < k)$ where Z is $N[0, 1+\sigma^2]$. Let W be another random variable which is also $N[0, 1+\sigma^2]$. We use a sequence of steps similar to the one above, only without the rotation:

$$
\begin{aligned}
\text{Prob}(Z < k) &= \text{Prob}(Z < k, -\infty < W < \infty) \\
&= \int\int_{z<k} \frac{1}{\sqrt{2\pi}\sqrt{1+\sigma^2}} e^{-z^2/2(1+\sigma^2)} \frac{1}{\sqrt{2\pi}\sqrt{1+\sigma^2}} e^{-w^2/2(1+\sigma^2)} dzdw \\
&= \frac{1}{2\pi(1+\sigma^2)}\int\int_{z<k} e^{-(z^2+w^2)/2(1+\sigma^2)} dzdw \\
&= \frac{1}{2\pi(1+\sigma^2)}\int\int_{z<k} re^{-r^2/2(1+\sigma^2)} drd\theta \\
&= \frac{1}{2\pi(1+\sigma^2)}\int_{\pi/2}^{3\pi/2}\int_0^{\infty} re^{-r^2/2(1+\sigma^2)} drd\theta + \\
&\quad \frac{1}{2\pi(1+\sigma^2)}\int_{-\pi/2}^{\pi/2}\int_0^{k/cos\theta} re^{-r^2/2(1+\sigma^2)} drd\theta \\
&= \frac{1}{2\pi(1+\sigma^2)}\int_{\pi/2}^{3\pi/2}\left[-(1+\sigma^2)e^{-r^2/2(1+\sigma^2)}\right]_0^{\infty} d\theta + \\
&\quad \frac{1}{2\pi(1+\sigma^2)}\int_{-\pi/2}^{\pi/2}\left[-(1+\sigma^2)e^{-r^2/2(1+\sigma^2)}\right]_0^{k/\cos\theta} d\theta \\
&= 1 - \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{\frac{-k^2}{2\cos^2\theta(1+\sigma^2)}}\,d\theta \qquad (8)
\end{aligned}
$$

Equations (7) and (8) are the same, so at this point we have completed our proof that $\text{Prob}(X + Y < k) = \text{Prob}(Z < k)$ when $k \geq 0$.

Now suppose that $k < 0$. Propositon 6 implies that for any normal random variable X with mean 0, $-X$ is also normally distributed with mean 0 and the same variance as X. Let $A = -X$, $B = -Y$, and $W = -Z$. Then A is $N[0, 1]$, B is $N[0, \sigma^2]$, and W is $N[0, 1+\sigma^2]$. So we have:

$$
\begin{aligned}
\text{Prob}(X + Y < k) &= \text{Prob}(-(X + Y) > -k) \\
&= \text{Prob}(A + B > -k) \\
&= 1 - \text{Prob}(A + B < -k) \\
&= 1 - \text{Prob}(W < -k) \quad \text{(because } -k > 0) \\
&= \text{Prob}(W > -k) \\
&= \text{Prob}(-Z > -k) \\
&= \text{Prob}(Z < k)
\end{aligned}
$$

At this point we have shown that $\text{Prob}(X+Y<k) = \text{Prob}(Z<k)$ for all k. Thus the random variables $X+Y$ and Z have the same cumulative density function. Z is $N[0, 1+\sigma^2]$, so $X+Y$ is also $N[0, 1+\sigma^2]$.

This completes our proof for the case that X is $N[0,1]$ and Y is $N[0,\sigma^2]$.

For the general case where X is $N[\mu_1, \sigma_1^2]$ and Y is $N[\mu_2, \sigma_2^2]$, let $A = (X-\mu_1)/\sigma_1$ and $B = (Y-\mu_2)/\sigma_1$. By Property 6, A is $N[0,1]$ and B is $N[0, \sigma_2^2/\sigma_1^2]$. Thus:

$$\begin{aligned}
\text{Prob}(X+Y<k) &= \text{Prob}(A+B<(k-\mu_1-\mu_2)/\sigma_1) \\
&= \int_{-\infty}^{(k-\mu_1-\mu_2)/\sigma_1} \frac{1}{\sqrt{2\pi}\sqrt{1+\sigma_2^2/\sigma_1^2}} e^{-x^2/2(1+\sigma_2^2/\sigma_1^2)} dx \\
&= (\text{apply the transformation } y = \sigma_1 x + \mu_1 + \mu_2) \\
&\quad \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} e^{-(y-(\mu_1+\mu_2))^2/2(\sigma_1^2+\sigma_2^2)} dy
\end{aligned}$$

This is the cumulative density function for $N[\mu_1+\mu_2, \sigma_1^2+\sigma_2^2]$, so we have our full result.

Corollary 1 *If X_i are independent $N[\mu,\sigma^2]$ for $i = 1 \ldots n$ then $\sum_{i=1}^{n} X_i$ is $N[n\mu, n\sigma^2]$.*

Proof:

This corollary follows immediately from Proposition 7.

Corollary 2 *If Y_i are independent $LN[\mu,\sigma^2]$ for $i = 1 \ldots n$ then $\prod_{i=1}^{n} Y_i$ is $LN[n\mu, n\sigma^2]$.*

Proof:

This corollary follows immediately from Definition 2 and Proposition 7.

Proposition 8 *The probability density function of* $LN[\mu, \sigma^2]$ *is:*

$$\frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log(x)-\mu)^2/2\sigma^2}$$

Proof:

Suppose X is $LN[\mu, \sigma^2]$. Thev $X = e^Y$ where Y is $N[\mu, \sigma^2]$. Then:

$$\begin{aligned}
\text{Prob}(X < k) &= \text{Prob}(e^Y < k) \\
&= \text{Prob}(Y < \log(k)) \\
&= \int_{-\infty}^{\log(k)} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} dy \\
&= (\text{apply the transformation } x = e^y,\ y = \log(x),\ dy = \tfrac{1}{x} dx) \\
&\quad \int_{-\infty}^{k} \frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log(x)-\mu)^2/2\sigma^2} dx
\end{aligned}$$

References

[1] John Norstad. Probability review.
http://www.norstad.org/finance, Sep 2002.