An Exact Equation for the Median Return

John Norstad

j-norstad@northwestern.edu
http://www.norstad.org

April 23, 2004
Updated: November 3, 2011

Abstract

A short derivation of the exact equation for computing the median or "annualized" return of a lognormal random walk asset, given the simply compounded average or "expected" return and standard deviation of the simply compounded returns.

1 An Exact Equation for the Median Return

Let Y be a lognormal random variable giving the ending value after one year of a \$1 investment in an asset. Let:

μ = the expected continuously compounded return of the asset.
σ = the standard deviation of the continuously compounded returns.

By Proposition 5 in reference [1] we have:

$$\begin{aligned} \mathrm{E}(Y) &= e^{\mu+\frac{1}{2}\sigma^2} \\ \mathrm{Var}(Y) &= e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \end{aligned}$$

Let:

r = the simply compounded expected return of the asset.
s = the standard deviation of the simply compounded returns.

Then:

$$\begin{aligned} 1+r &= \mathrm{E}(Y) \\ &= e^{\mu+\frac{1}{2}\sigma^2} \\ s^2 &= \mathrm{Var}(Y) \\ &= e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \\ &= \left(e^{\mu+\frac{1}{2}\sigma^2}\right)^2(e^{\sigma^2}-1) \\ &= (1+r)^2(e^{\sigma^2}-1) \end{aligned}$$

We can solve these equations for σ^2 and μ:

$$\begin{aligned} \sigma^2 &= \log\left(\frac{s^2}{(1+r)^2}+1\right) \\ \mu &= \log(1+r)-\frac{1}{2}\sigma^2 \\ &= \log(1+r)-\frac{1}{2}\log\left(\frac{s^2}{(1+r)^2}+1\right) \\ &= \log(1+r)-\log\left(\sqrt{\frac{s^2}{(1+r)^2}+1}\right) \\ &= \log\left(\frac{1+r}{\sqrt{\frac{s^2}{(1+r)^2}+1}}\right) \\ &= \log\left(\frac{(1+r)^2}{\sqrt{s^2+(1+r)^2}}\right) \end{aligned}$$

Let:

$$a = \text{the median yearly simply compounded return}$$

Then:

$$\begin{aligned} a &= e^{\mu} - 1 \\ &= e^{\log\left(\frac{(1+r)^2}{\sqrt{s^2+(1+r)^2}}\right)} - 1 \\ &= \frac{(1+r)^2}{\sqrt{s^2+(1+r)^2}} - 1 \end{aligned}$$

This is our exact equation for the median return.

If we take the Taylor series expansion of this equation about $s = 0$, the first three terms give the following approximation:

$$a \approx r - \frac{\frac{1}{2}s^2}{1+r}$$

This is similar to, but a bit more accurate than, the frequently seen approximation:

$$a \approx r - \frac{1}{2}s^2$$

References

[1] John Norstad. The normal and lognormal distributions. http://www.norstad.org/finance, Feb 1999.