

# Asset Allocation and Portfolio Survival

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## **Abstract**

We examine the impact of asset allocation on the problem of outliving your money in retirement. Given a real (inflation-adjusted) withdrawal rate, we run a large number of Monte Carlo simulations to determine the asset allocation which maximizes the 30 year survival rate. We see how the composition of this optimal portfolio changes as the withdrawal rate changes.

## 1 The Model

We use the random walk model developed in [2]:

$$\begin{aligned}\frac{\Delta s}{s} &= e^{\mu\Delta t + \sigma\Delta X} - 1 + \frac{k\Delta t}{s} \quad \text{where } \Delta X \text{ is } N[0, \Delta t] \\ s(t + \Delta t) &= s(t)e^{\mu\Delta t + \sigma\Delta X} + k\Delta t\end{aligned}$$

$t$  is the time in years.

$\Delta t$  is a time interval.

$s(t)$  is the portfolio value at time  $t$ .

$\mu$  is the expected continuously compounded yearly rate of return for the portfolio.

$\sigma$  is the standard deviation of the continuously compounded yearly returns.

$k$  is a constant yearly amount added to ( $k > 0$ ) or subtracted from ( $k < 0$ ) the portfolio in even installments at the end of each time interval  $\Delta t$ . In this paper we always have  $k < 0$ .

We use the constant  $t = 30$  to model 30 years of portfolio withdrawals and  $\Delta t = 1/12 = 0.08333\dots$  to model monthly withdrawals from the portfolio. We use  $s(0) = \$100$  for the initial portfolio value.

Given values for the parameters  $\mu$ ,  $\sigma$ , and  $k$ , we use a computer program to run 100,000 Monte Carlo simulations under this model of portfolio growth over 30 years. The *survival rate* is the percentage of these simulations with ending values  $s(30) \geq 0$ .

We consider feasible portfolios  $\langle c, b, s \rangle$  with asset allocation

$$\begin{aligned}c &= \text{percent cash} \geq 0 \\ b &= \text{percent bonds} \geq 0 \\ s &= \text{percent stocks} \geq 0 \\ 1 &= c + b + s\end{aligned}$$

Given such a portfolio  $\langle c, b, s \rangle$ , we estimate  $\mu$  and  $\sigma$  using historical time series data from 1926 through 1994.<sup>1</sup>

$$\begin{aligned}C &= \text{90 day US Treasury bills} \\ B &= \text{20 year US Treasury bonds} \\ S &= \text{S\&P 500 stocks} \\ I &= \text{Consumer price index}\end{aligned}$$

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<sup>1</sup>The data is from Table 2-4 in [1].

We assume constant investment expenses of 35 basis points and construct a time series  $N$  for the nominal return of the portfolio after expenses:

$$N = cC + bB + sS - 0.0035$$

We adjust for inflation by converting to real returns relative to the CPI:

$$R = \frac{N - I}{1 + I}$$

We convert to continuous compounding and take the mean and standard deviation:

$$\begin{aligned}\mu &= E(\log(1 + R)) \\ \sigma &= \text{Stdev}(\log(1 + R))\end{aligned}$$

Note that this study is adjusted for inflation. We keep our withdrawal amount  $k$  constant and in our parameter estimation we convert from nominal to real rates of return. For our purpose of computing survival rates, this is equivalent to using nominal rates of return and varying  $k$  each month to adjust the withdrawal amount for inflation. We omit the trivial proof.

## 2 The Results

The figures and tables on the following pages show the 30 year portfolio survival rates for real withdrawal rates of 3%, 4%, 5%, 6%, and 7%, with the portfolios varied over the feasible set in increments of 10% in each dimension.

At a 3% withdrawal rate the 30 year survival rates are very good. The maximum survival rate of 98% is achieved by a number of portfolios. All of the optimal portfolios contain at least 20% cash, at least 20% stocks, and at most 40% bonds. Note that asset allocation makes a big difference. For example, an all bond portfolio has a survival rate of only 83%, and an all cash portfolio has a survival rate of only 84%.

At a 4% withdrawal rate the 30 year survival rates are considerably less attractive than they are at 3%, but still not bad. The maximum survival rate drops from 98% to 88%. The optimal portfolios contain at most 10% cash, between 20% and 40% bonds, and at least 60% stocks.

At a 5% withdrawal rate the maximum survival rate drops to 75% and the optimal portfolios are almost all stocks – at least 90% of the portfolio must be in stocks to achieve this survival rate.

At the 6% and 7% withdrawal rates the maximum survival rates drop significantly – to 63% and 51% respectively. In both cases the optimal portfolio is all stocks.

These results confirm common sense. Higher withdrawal rates have lower maximum survival rates. At lower withdrawal rates more conservative portfolios are optimal. The priority at these lower rates is to reduce volatility to preserve capital. At higher withdrawal rates more aggressive portfolios are optimal. We need to generate higher returns at these rates to support the withdrawals, at the cost of increased risk of failure due to the higher volatility.

Note that the optimal portfolios contain some stocks even at the lower withdrawal rates.

In this paper we ignore the possible desire of a retiree to leave a bequest for his or her heirs. If we took this desire into account, for example, at a withdrawal rate of 3%, a retiree might prefer a more aggressive portfolio in an attempt to increase the size of the estate, at the cost of a decreased survival probability.

The results required a large amount of computation:

$$\begin{aligned}
 & 12 \text{ simulated months per year} \\
 & \times 30 \text{ years per simulation} \\
 & \times 100,000 \text{ simulations per portfolio} \\
 & \times 66 \text{ portfolios per withdrawal rate} \\
 & \times 5 \text{ withdrawal rates} \\
 & = 11,880,000,000 \text{ simulated months.}
 \end{aligned}$$

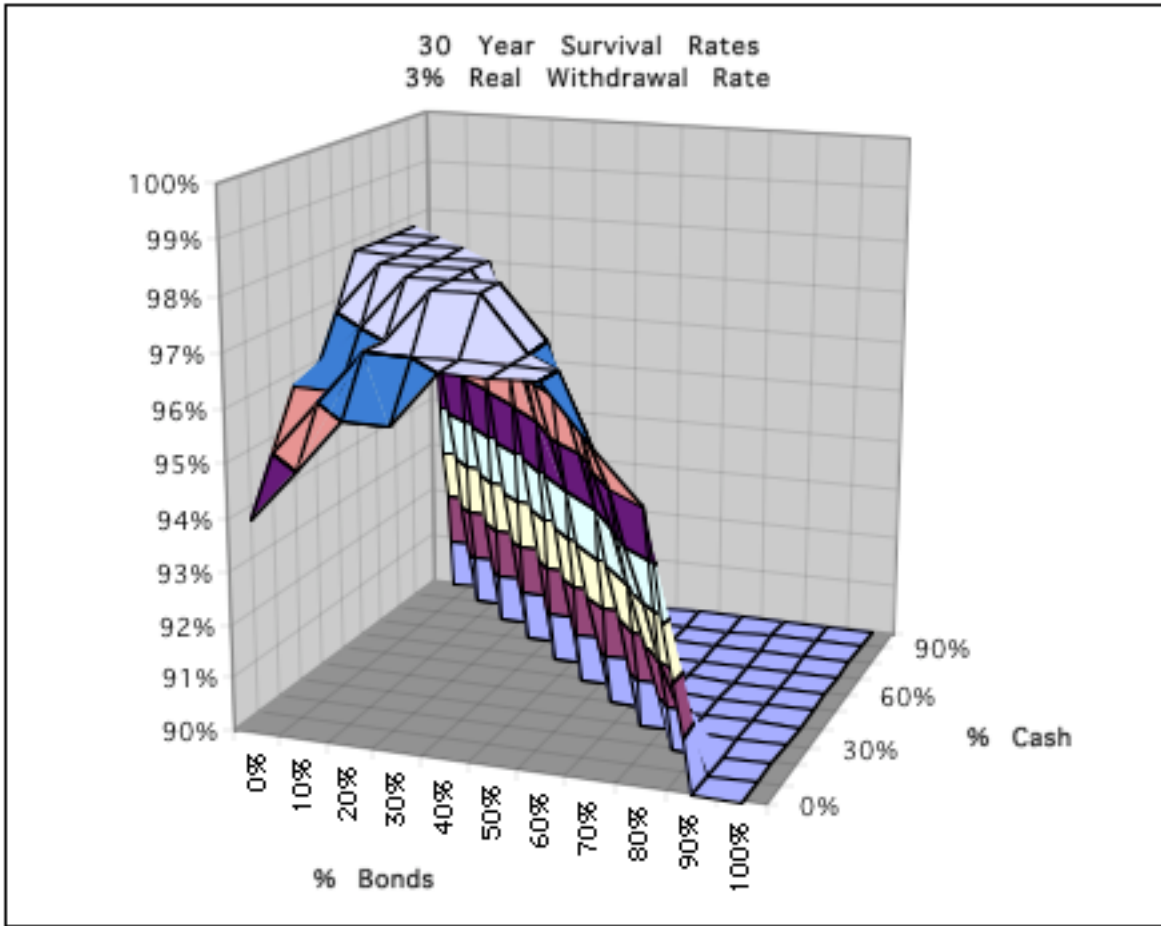
The computations took approximately six hours to perform on a 233 MHz Apple iMac. It's interesting that just a few years ago this kind of study would not have been feasible on low-end personal computers.<sup>2</sup>

In the first draft of this paper we attempted to cut corners to save time by decreasing the number of simulations per portfolio from 100,000 to 10,000. We discovered that with only 10,000 simulations the resulting survival rates were not accurate to more than about a percent or two. Even with 100,000 simulations, we do not feel that it would be justified to publish the resulting survival rate percentages with more precision than 0 digits after the decimal point.

If we were to attempt to model more than three asset classes, the computations would quickly get out of hand. For example, with four asset classes instead of three, there are 286 portfolios instead of 66 when we vary the asset allocation by 10% in each dimension. In this case, we would probably want to first compute the efficient portfolios over the feasible set, then perform the simulations over just the efficient portfolios. This is adequate for the purpose of computing optimal portfolios and maximum survival rates because if portfolio A is more efficient than portfolio B, then A has a higher survival rate than B under the random walk model.

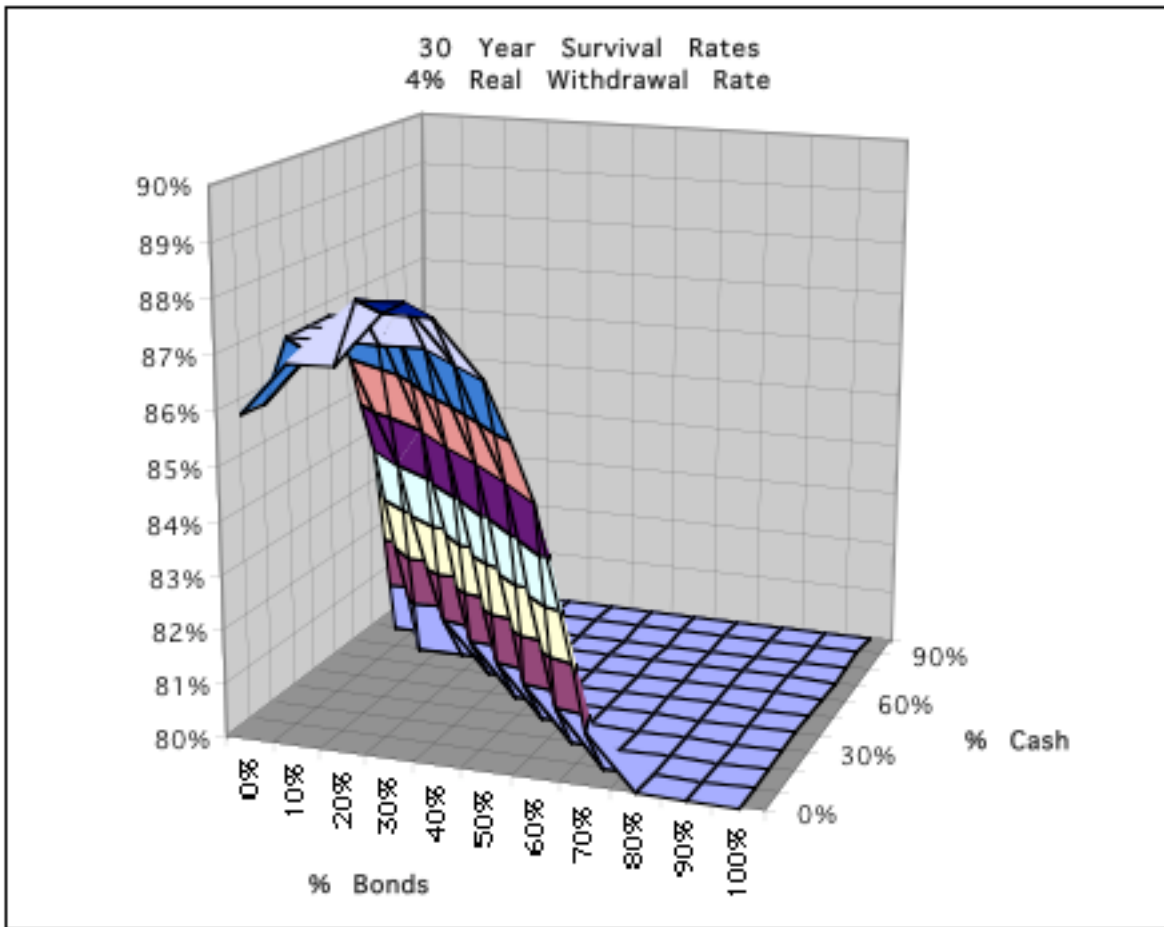
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<sup>2</sup>These computations were done back in 1999. Typical personal computers in 2005 are much faster!



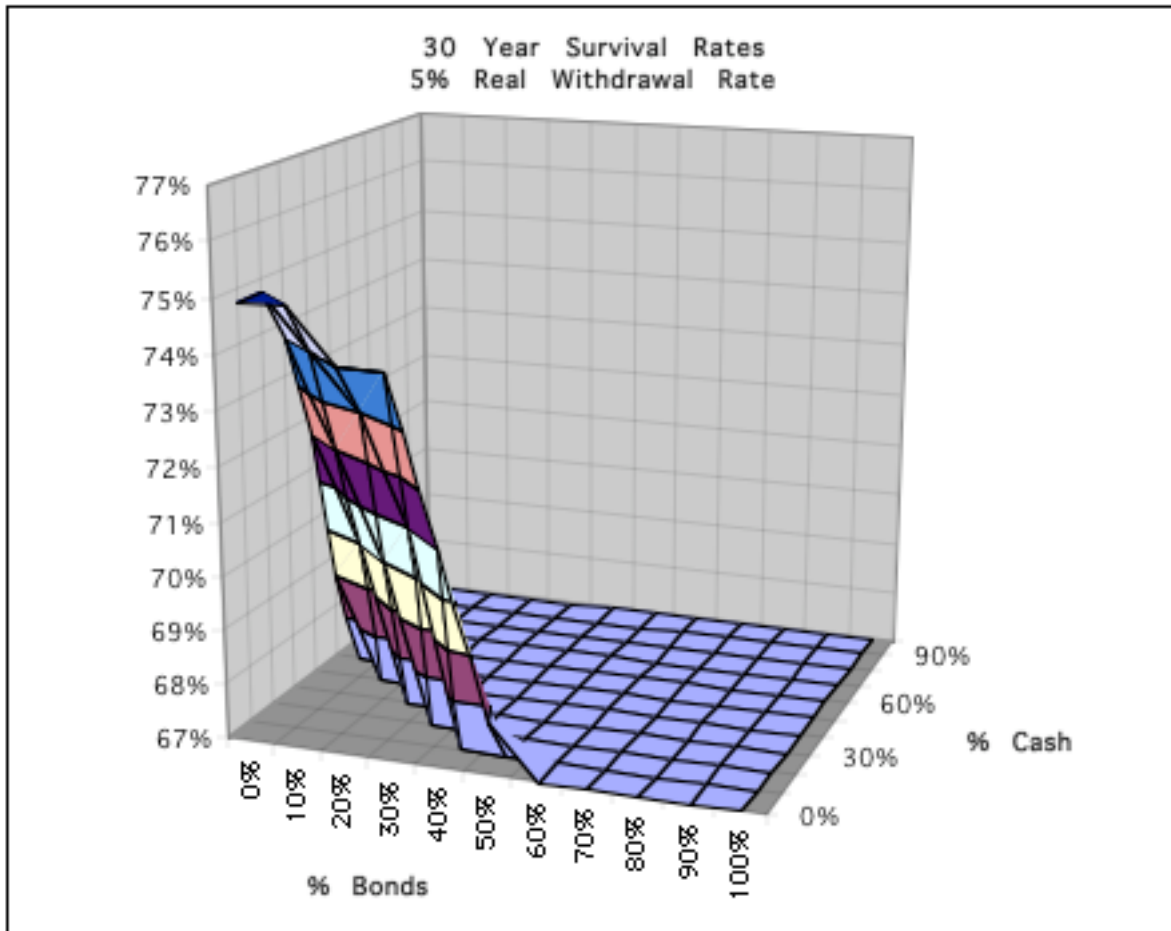
% Cash	% Bonds										
	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
0%	94%	95%	96%	96%	97%	97%	97%	96%	95%	90%	83%
10%	95%	96%	97%	97%	97%	97%	97%	95%	92%	83%	
20%	96%	96%	97%	<b>98%</b>	<b>98%</b>	97%	96%	93%	84%		
30%	96%	97%	<b>98%</b>	<b>98%</b>	<b>98%</b>	97%	94%	85%			
40%	97%	<b>98%</b>	<b>98%</b>	<b>98%</b>	97%	94%	86%				
50%	<b>98%</b>	<b>98%</b>	<b>98%</b>	<b>98%</b>	95%	87%					
60%	<b>98%</b>	<b>98%</b>	<b>98%</b>	96%	87%						
70%	<b>98%</b>	<b>98%</b>	96%	87%							
80%	<b>98%</b>	96%	87%								
90%	96%	86%									
100%	84%										

Figure 1: 3% Real Withdrawal Rate



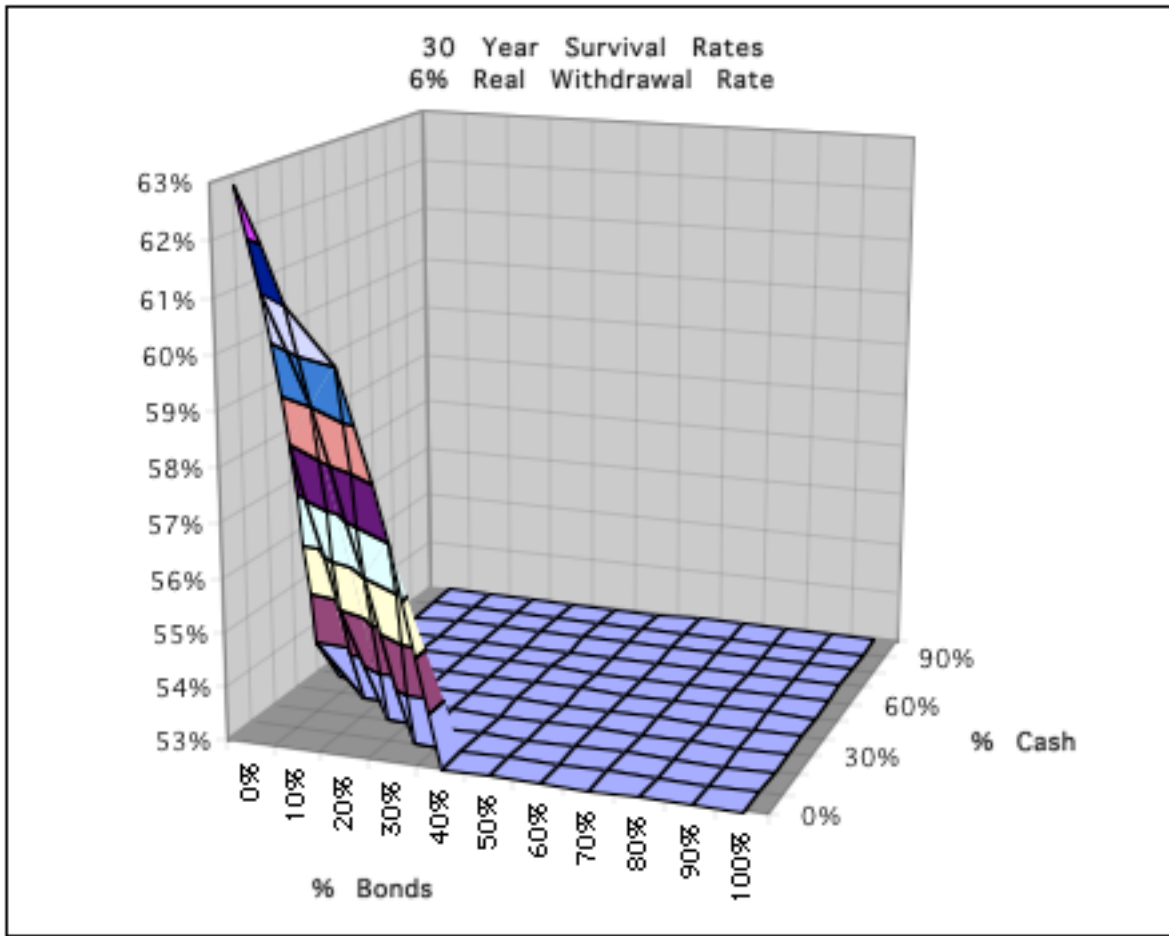
% Cash	% Bonds										
	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
0%	86%	87%	87%	<b>88%</b>	<b>88%</b>	87%	85%	81%	74%	63%	49%
10%	86%	87%	<b>88%</b>	<b>88%</b>	87%	85%	81%	74%	62%	48%	
20%	87%	87%	87%	87%	85%	81%	74%	61%	46%		
30%	87%	87%	87%	85%	81%	74%	60%	44%			
40%	87%	87%	85%	81%	73%	58%	40%				
50%	86%	84%	81%	72%	56%	37%					
60%	84%	80%	71%	54%	32%						
70%	78%	69%	50%	28%							
80%	66%	45%	22%								
90%	39%	17%									
100%	12%										

Figure 2: 4% Real Withdrawal Rate



% Cash	% Bonds										
	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
0%	<b>75%</b>	<b>75%</b>	74%	74%	71%	68%	62%	54%	43%	31%	21%
10%	<b>75%</b>	74%	73%	71%	67%	62%	53%	41%	29%	18%	
20%	74%	72%	70%	67%	61%	51%	39%	25%	16%		
30%	72%	70%	66%	59%	49%	36%	22%	12%			
40%	69%	65%	58%	47%	33%	18%	9%				
50%	63%	56%	44%	29%	14%	6%					
60%	54%	41%	25%	11%	4%						
70%	38%	21%	7%	2%							
80%	17%	5%	1%								
90%	3%	0%									
100%	0%										

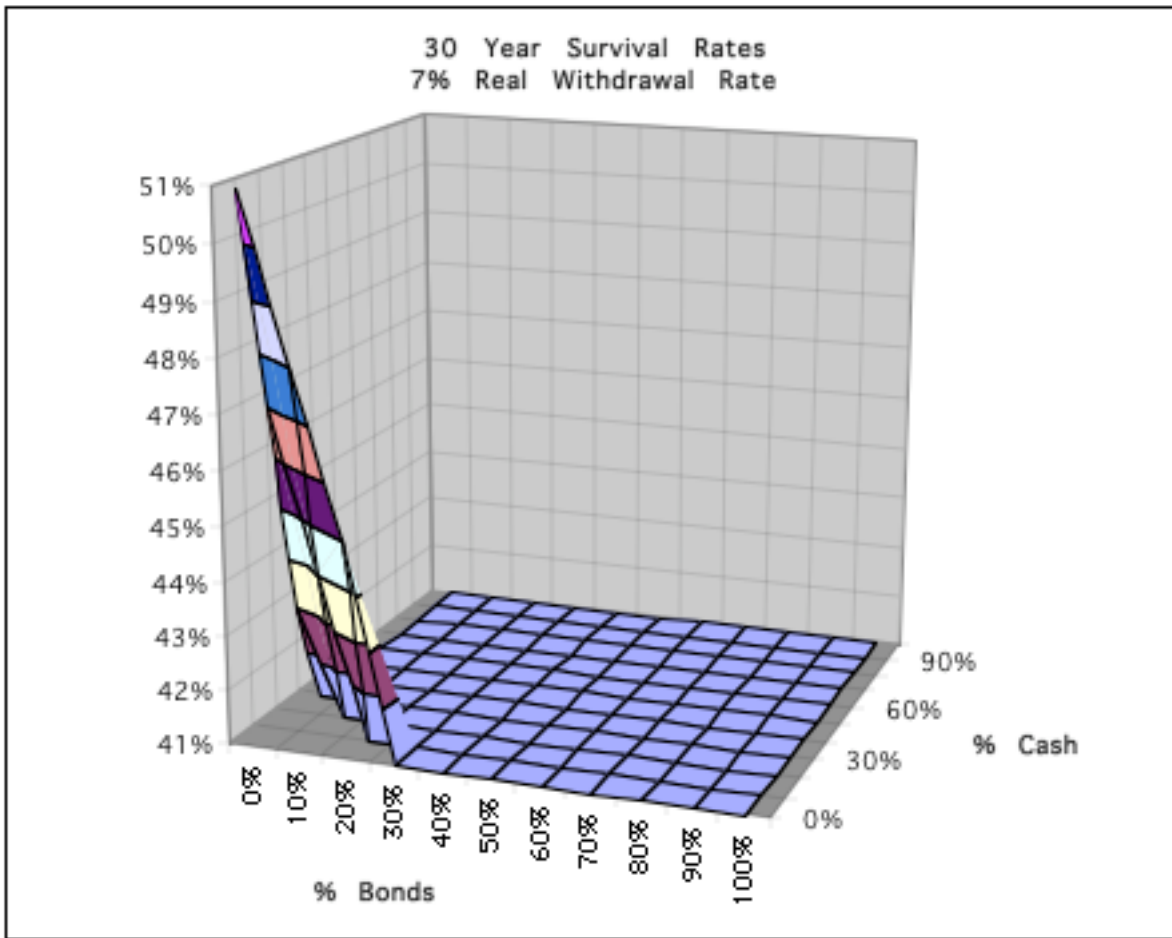
Figure 3: 5% Real Withdrawal Rate



% Cash	% Bonds										
	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
0%	<b>63%</b>	61%	60%	57%	53%	46%	39%	29%	19%	12%	7%
10%	61%	59%	56%	52%	45%	37%	26%	17%	9%	5%	
20%	58%	55%	51%	43%	35%	24%	14%	7%	4%		
30%	54%	49%	42%	32%	21%	11%	5%	2%			
40%	47%	40%	30%	18%	9%	3%	1%				
50%	38%	28%	16%	6%	2%	1%					
60%	25%	13%	4%	1%	0%						
70%	11%	3%	0%	0%							
80%	2%	0%	0%								
90%	0%	0%									
100%	0%										

Figure 4: 6% Real Withdrawal Rate





% Cash	% Bonds										
	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
0%	<b>51%</b>	48%	45%	41%	35%	28%	20%	13%	7%	3%	2%
10%	47%	44%	40%	34%	26%	18%	11%	5%	2%	1%	
20%	43%	39%	32%	25%	16%	9%	4%	1%	1%		
30%	37%	31%	23%	14%	7%	2%	1%	0%			
40%	29%	21%	12%	5%	1%	0%	0%				
50%	19%	11%	4%	1%	0%	0%					
60%	9%	3%	0%	0%	0%						
70%	2%	0%	0%	0%							
80%	0%	0%	0%								
90%	0%	0%									
100%	0%										

Figure 5: 7% Real Withdrawal Rate

## References

- [1] Zvi Bodie and Robert C. Merton. *Finance*. Prentice-Hall, preliminary edition, 1998.
- [2] John Norstad. Financial planning using random walks. <http://www.norstad.org/finance>, Feb 1999.